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FIELD THEORY

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Abstract
The following 5 lectures introduce various key concepts in perturbative quantum field theory. The discussion is based on rather detailed treatments of examples rather than on expositions of a more general nature.

1 LECTURE 1. VERY ELEMENTARY FREE FIELD THEORY

1.1 Introductory remarks

To motivate the study of quantum field theory in the late 1990'ies is not a difficult task: The Standard Model works amazingly well; field theory is obviously just the right language for dealing with the laws of nature at the most fundamental level that can be experimentally investigated now. It was much less apparent in the 1960'ies that this should be so. At that time field theories of the strong interactions based on mesons and baryons, appeared complicated and untractable, in perturbation theory at least. The breakthrough came with the realization that quarks (and later gluons), not hadrons should form the staring point of simple descriptions, and that deep inelastic scattering and later asymptotic freedom indicated that in some situations at least, the interactions could indeed be described by perturbation theory. Effective field theories based on hadrons have become phenomenologically popular again, but that is a different story not covered here.

Apart from the fact that it works, there are some more fundamental hints that quantum field theory is the right language. The notion of a single particle wave function, for example, so useful in non relativistic quantum mechanics, is an untenable concept in relativistic physics. Physically this is because a very precise determination of the position of a particle requires very short wavelength radiation, hence the presence of a quantum with sufficient energy to pair produce a particle-antiparticle pair, so that the single particle description breaks down. A crucial experimental hint comes from the remarkable identity of like elementary particles. The deep identity of all electrons is not dealt with at all in non relativistic quantum mechanics, but is directly related to the concept of an electron quantum field.

It used to be the case that most people regarded renormalizability of a quantum field theory as a necessary condition for the theory to make sense. And indeed in these lectures we shall deal only with renormalizable field theories of which of course the Standard Model is the prime example. However the attitude has changed perhaps in response in part to the impact of string theory. String theory (whether related to the real world or not) is not a field theory, yet at low energies it cannot help looking like one, and in fact one containing non renormalizable ingredients, such as supergravity. The view has emerged that the characteristics of quantum field theory is an inevitable consequence of quantum theory combined with relativity - plus some other ideas such as locality in one form or another. For a recent account of this see the talk by Weinberg [1] further explained in his text book [2].

The attitude in these lectures is going to be that it will not be possible to provide a systematic introduction to the subject in the 5 lectures, so the emphasis will be on somewhat detailed illustrations of key concepts by way of examples, rather than by way general expositions. Several details were left as exercises when I gave the lectures, but here some of them are incorporated in the text.

In preparing these lectures I have mostly made use of the references given at the end, notably our Copenhagen lecture notes [3, 4] and the textbook by Peskin and Schroeder [5]. Numerous other excellent text books exist.

1.2 Scalar field theory

It turns out that a powerful way of realizing the requirements to a theory coming from relativity and quantum mechanics, consists in formulating the theory in terms of the *action*. This is particularly true in perturbation theory, and we shall deal with field theory almost exclusively in terms of perturbation theory. The action is a Lorentz scalar and so is the *lagrangian density.* The idea of *locality* may be made precise by requiring that the lagrangian density be a *local* function of the field(s) and a finite number of derivatives; here we shall consider first order dependencies only. Thus with (x) being coordinates of a space time point, $\phi(x)$ an associated field (taken scalar for simplicity here), the lagrangian density $\mathcal{L}$ and the action S are given by

$$\begin{aligned} \mathcal{L} &= \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\ S &= \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \end{aligned} \tag{1}$$

A few words about notation. We shall use units so that $c = \hbar = 1$. Coordinates of an event are denoted

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \vec{x})$$

Likewise the 4-momentum 4-vector is denoted

$$p^\mu = (E, \vec{p})$$

For any 4-vector, we go between lower and upper indices using the metric

$$\eta^{\mu\nu} = \text{diagonal}(+1, -1, -1, -1)$$

thus $p_\mu = (E, -\vec{p})$. Under Lorentz transformation we shall write

$$p^\mu \to p'^\mu = \Lambda^\mu{}_\nu p^\nu \tag{2}$$

with summation over repeated indices understood. The Lorentz transformation coefficients are restricted by the requirement that $a_\mu b^\mu$ is an invariant for a, b 4-vectors. For example

$$p^2 = p_\mu p^\mu = E^2 - \vec{p}^2 = m^2 \tag{3}$$

where m is the (rest) mass of the particle with 4-momentum p. The derivatives

$$\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} \\ \partial^\mu &\equiv \frac{\partial}{\partial x_\mu} \end{aligned} \tag{4}$$

produce 4-vectors with indices as indicated when acting on scalar fields.

Classical field theory selects the classical field $\phi_{cl}(x)$ satisfying the *equations of motion* (plus in practice certain boundary conditions). These equations are obtained from the condition that the action evaluated at the classical field is stable with respect to small variations of the field: $\phi(x) = \phi_{cl}(x) + \delta\phi(x) \Rightarrow \delta S = 0$. This condition is rather easily seen to give rise to the *Euler-Lagrange* equations of motion:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \tag{5}$$

This kind of equation generalizes trivially to the case with more than one field (-component).

For a free field we want quanta having 4-momentum p^μ satisfying the mass shell condition $p^2 = m^2$. That suggests that we should get the (free) wave equation for the field by the substitution $p^\mu \to i\partial^\mu$, familiar from quantum mechanics, leading to the *Klein-Gordon equation*

$$-\partial_\mu \partial^\mu \phi = m^2 \phi \tag{6}$$

A lagrangian giving rise to that equation as a result of the Euler-Lagrange equations (5) is

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{7}$$

Namely

$$\begin{aligned} \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi)} &= \frac{2}{2}\partial^\mu \phi \\ &\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi)} \right) = \partial_\mu \partial^\mu \phi \\ \frac{\partial \mathcal{L}_0}{\partial \phi} &= -m^2 \phi \end{aligned} \tag{8}$$

A complete set of solutions to the Klein-Gordon equation consists of the plane waves

$$\phi_{\vec{p}}(x) = e^{-ip_\mu x^\mu} = e^{-i(p^0 t - \vec{p}\vec{x})} \tag{9}$$

provided $p^2 = m^2$ or

$$p^0 = \pm\sqrt{\vec{p}^2 + m^2} \tag{10}$$

Here we see the celebrated mystery of “negative energy” solutions, a serious problem in the old wave function interpretation of the field, but a problem completely solved in the quantum field interpretation of it. Let us write a general solution to the equations of motion as a linear combination of the complete set, and let us write the “negative energy solutions” in the form e^{+ikx} with

$$k^0 = +\sqrt{\vec{k}^2 + m^2}$$

Thus

$$\phi(x) = \sum_{\vec{k}} \left(a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{+ikx} \right) \tag{11}$$

Here we use the short hand notation

$$\sum_{\vec{k}} \equiv \int \frac{d^3k}{2k^0 (2\pi)^3}$$

which is a Lorentz invariant integration measure. In the classical theory the coefficients $a(\vec{k}), a^\dagger(\vec{k})$ would be arbitrary complex coefficients, and reality of ϕ would imply that the two are each other’s complex conjugate.

In the operator description of quantum field theory on the other hand, $\phi(x)$ is treated as an operator - a time dependent one in the Heisenberg picture. And the coefficients $a(\vec{k}), a^\dagger(\vec{k})$ are now annihilation and creation operators satisfying harmonic oscillator type commutation relations

$$\begin{aligned} [\alpha(\vec{k}), a^\dagger(\vec{k}')] &= \delta_{\vec{k}\vec{k}'} \\ [a(\vec{k}), a(\vec{k}')] &= 0 = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] \end{aligned} \tag{12}$$

where we have written

$$\delta_{\vec{k},\vec{k}'} \equiv 2k^0 (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

so that

$$\sum_{\vec{k}} \delta_{\vec{k}\vec{k}'} = 1$$

We shall not derive these commutation relations using standard canonical quantization. But a hint to what goes on is obtained by working out the hamiltonian

$$\begin{aligned}
H &= \int d^3x \mathcal{H} \\
\mathcal{H} &= \dot{\phi}\pi - \mathcal{L} \\
\pi(x) &\equiv \frac{\partial \mathcal{L}}{\partial(\dot{\phi})} = \dot{\phi}(x) \Rightarrow \\
\mathcal{H} &= \frac{1}{2}((\dot{\phi})^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2) \qquad (13)
\end{aligned}$$

One finds

$$\begin{aligned}
H &= \sum_{\vec{k}} k^0 (N_{\vec{k}} + \frac{1}{2}) \\
N_{\vec{k}} &= a^\dagger(\vec{k}) a(\vec{k}) \qquad (14)
\end{aligned}$$

the familiar expression for a sum of many harmonic oscillators, one for each possible 3-momentum. This expression demonstrates that the hamiltonian is positive definite: the possible eigenvalues of $N_{\vec{k}}$ are $0, 1, 2, \ldots$. The Hilbert space of our free quantum field theory is the Fock space spanned by states of definite particle number:

$$\begin{aligned}
|0\rangle & \quad \text{vacuum} \\
a(\vec{k})|0\rangle &= 0 \\
|\vec{k}\rangle = a^\dagger(\vec{k})|0\rangle & \quad \text{1-particle states} \\
|\vec{k}_1, \vec{k}_2\rangle = a^\dagger(\vec{k}_1) a^\dagger(\vec{k}_2)|0\rangle & \quad \text{2-particle states} \\
\vdots & \quad \vdots \qquad (15)
\end{aligned}$$

These particles are *bosons* by virtue of the commutation relations (12):

$$|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle \qquad (16)$$

1.2.1 The scalar propagator I

A simple interaction term which we shall study later is obtained by writing

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4 \qquad (17)$$

with the equation of motion

$$(\partial_\mu \partial^\mu + m^2)\phi = -\frac{\lambda}{3!}\phi^3 \qquad (18)$$

For now we prefer to study the simpler case of en external source $J(x)$ for the scalar field

$$\mathcal{L}_J = \mathcal{L}_0 + J(x)\phi(x) \qquad (19)$$

with the equation of motion

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = J(x) \qquad (20)$$

We can solve this classical equation thereby obtaining the field $\phi(x)$ resulting from the "transmitter source" $J(x)$. The key to solving this problem is the *propagator* $D(x)$ satisfying

$$(\partial^2 + m^2)D(x) = -i\delta^4(x) \tag{21}$$

by means of which we would clearly get

$$\phi(x) = i\int d^4y D(x-y)J(y) \tag{22}$$

as is immediately verified. It seems easy to find the propagator since

$$D(x) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2-m^2} \tag{23}$$

evidently satisfies the defining equation (21). However, a subtlety remains, since the propagator as we have written down does not make mathematical sense due to the presence of the singularity in the integrand at $k^2 = m^2$. In quantum field theory Feynman realized that the correct cure was to use the *Feynman prescription*

$$D_F(x) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2-m^2+i\epsilon} \tag{24}$$

where ϵ is a small positive number that has to be taken to zero at the end of the calculation. As we shall see, this prescription is linked to a particular choice of boundary conditions. Actually, in a classical context one would usually prefer different boundary conditions from the ones in the quantum theory as we shall see in lecture 4. Evidently classical solutions obtained using different propagators will differ from each others by solutions to the homogeneous, i.e. the free Klein-Gordon equation. In other words the difference between particular solutions consists of "radiation". Classically we might impose the boundary condition that no radiation should be present before the source is turned on. That would give rise to the so called causal propagator.

Apart from that subtlety, we have learned an easy and powerful way of constructing the propagator: we naively invert the differential operator in the relevant classical wave equation, and then we insert the Feynman $+i\epsilon$.

1.2.2 The scalar propagator II

Now let us argue for eq.(24) using quite a different, quantum picture of what the propagator is. In fact we want to show that

$$D_F(x) = \langle 0|T\{\phi(x)\phi(0)\}|0\rangle \tag{25}$$

Remember that the field *operator* $\phi(x)$ may be thought of as a combination of terms that create, respectively annihilate 1 particle at the point x. The object in question represents the amplitude for *first* creating one particle at $x = 0$ and *subsequently* annihilating one at x. This amplitude is therefore the amplitude for a particle to *propagate* from 0 to x. We have emphasized the *time ordering* implied: If $t = x^0 > 0$ the situation is as described. If on the other hand $x^0 < 0$ the *time ordering operator* T reverses the order of the two field operators, so that always creation comes before annihilation. That is the boundary condition crucial in the Feynman $i\epsilon$ prescription.

We leave the proof of eq.(25) as an exercise, using the following hints: Introduce the step function

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0 \end{cases} \tag{26}$$

then write

$$\langle 0|T\{\phi(x)\phi(0)\}|0\rangle = \theta(x^0)\langle 0|\{\phi(x)\phi(0)|0\rangle + \theta(-x^0)\langle 0|\phi(0)\phi(x)|0\rangle \tag{27}$$

Use the plane wave expansion of the free field, eq.(11) to get for the first term

$$\theta(x^0)\sum_{\vec{k},\vec{k}'}\langle 0|a(\vec{k})e^{-ikx}a^\dagger(\vec{k}')e^{ik'\cdot 0}|0\rangle$$
$$= \theta(x^0)\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}e^{-iE_{\vec{k}}x^0}e^{i\vec{k}\cdot\vec{x}} \tag{28}$$

where $E_{\vec{k}} = +\sqrt{\vec{k}^2 + m^2}$. Verify that

$$\theta(x^0)e^{-iE_{\vec{k}}x^0} = i\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\frac{e^{-ik^0x^0}}{k^0 - E_{\vec{k}} + i\epsilon} \tag{29}$$

by showing how to close the integration countour in the upper (lower) complex k^0 half-plane by a large semicircle, according as $x^0 < 0$ ($x^0 > 0$), and then by using residue calculus. Carry out a suitable similar treatment of the second term, and finally collect pieces.

1.2.3 The path integral. The scalar propagator III

So far we have used the operator formulation of quantum field theory. Let us briefly mention the path integral formulation. It gives rise to the easiest derivation of Feynman rules in the general case. It gives the cleanest treatment of perturbative quantization of non-Abelian gauge theories. It provides the best intuitive approach to renormalization theory. And it forms the starting point for non-perturbative lattice simulations of quantum field theories. We refrain completely from proving the "equivalence" of the two formulations. We restrict ourselves to providing a simple example of how the formalism works.

For a simple scalar theory the equivalence between the two formulations may be given more precisely as the following equivalent expressions for the so-called Greens functions:

$$\langle 0|T\{\phi(x_1)\phi(x_2)...\phi(x_n)|0\rangle = N^{-1}\int \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]}\phi(x_1)\phi(x_2)...\phi(x_n)$$
$$N = \int \mathcal{D}\phi e^{\frac{i}{\hbar}S[\phi]} \tag{30}$$

In the first equality, the fields $\phi(x)$ on the left hand side are Heisenberg *operators.* On the right hand side we have the path integral with an integration measure $\mathcal{D}\phi$ over "all classical field histories". The action $S[\phi]$ is evaluated at a particular such classical field, but in general one *not* satisfying the classical field equations. Also the objects $\phi(x_1)$ etc. are the values of that classical field at the space time points x_1 etc. The meaning of the path integral is defined in terms of slicing up space-time, replacing it by a mesh of discrete points to be made denser and denser in some suitable limiting procedure. As mentioned we will not prove this result but restrict ourselves to illustrating it in the case $n = 2$, in other words show how the path integral gives rise to the same propagator as before. So we shall demonstrate that

$$D_F(x_1 - x_2) = N^{-1}\int \mathcal{D}\phi e^{i\int d^4x\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 + i\epsilon\phi^2]}\phi(x_1)\phi(x_2) \tag{31}$$

where we have introduced a "convergence factor" $e^{-\epsilon\frac{1}{2}\phi^2}$ making sure that the oscillating integrand is damped for large fields.

It is very convenient first to introduce another object the (free) generating functional defined by

$$Z_0[J] = \int \mathcal{D}\phi e^{i\int d^4x\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 + i\epsilon\phi^2] + i\int d^4x J(x)\phi(x)}$$
$$N = Z_0[0] \tag{32}$$

The index 0 is to remind us that this is the *free* generating functional. From this generating functional, Greens functions, including the 2-point function namely the propagator, may be easily obtained by *functional differentiation.* We define the functional derivative by

$$\frac{\delta J(x)}{\delta J(y)} \equiv \delta^4(x-y) \tag{33}$$

Then

$$\frac{\delta}{\delta J(x_1)} e^{i\int d^4x J(x)\phi(x)} = i\phi(x_1) e^{i\int d^4x J(x)\phi(x)} \tag{34}$$

Hence

$$\int \mathcal{D}\phi e^{iS} \phi(x_1)\phi(x_2) = \frac{\delta}{i\delta J(x_1)} \frac{\delta}{i\delta J(x_2)} Z_0[J]|_{J\equiv 0} \tag{35}$$

We want to show that this is the propagator (up to normalization by $Z_0[0]$). We can do the integral since it is of *Gaussian type.* The integrand is the exponential of something which we rewrite as

$$\begin{aligned} &\int d^4x\{\frac{1}{2}[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 + i\epsilon\phi^2] + J\phi\} \\ &= \int d^4x\{-\frac{1}{2}\phi(\partial^2 + m^2 - i\epsilon)\phi + J\phi\} \end{aligned} \tag{36}$$

First consider 1-dimensional Gaussian integrals

$$\begin{aligned} \int_{-\infty}^{\infty} d\phi e^{-A\phi^2} &= \sqrt{\frac{\pi}{A}} \\ \int_{-\infty}^{\infty} e^{-A\phi^2 + iJ\phi} &= \int_{-\infty}^{\infty} e^{-A\left\{\left(\phi - i\frac{J}{2A}\right)^2 + \frac{J^2}{4A^2}\right\}} \\ &= \text{const.} e^{-\frac{J^2}{4A}} \\ \text{const.} &= \sqrt{\frac{\pi}{A}} \end{aligned} \tag{37}$$

The reader is invited to generalize this to the finite dimensional case with the result (A is a symmetric or hermitean $n \times n$ matrix with positive eigenvalues)

$$\begin{aligned} \int \prod d\phi_i e^{-\phi_l A_{lm}\phi_m + iJ_k\phi_k} &= \text{const.} e^{-\frac{1}{4}J_i(A^{-1})_{ij}J_j} \\ \text{const.} &= \sqrt{\frac{\pi^n}{\det A}} \end{aligned} \tag{38}$$

We naively generalize this to the functional case and obtain

$$Z_0[J] = Z_0[0]\exp\{\frac{i}{2}\int d^4x d^4y J(x)(\partial^2 + m^2 - i\epsilon)^{-1}(x-y)J(y)\} \tag{39}$$

But from our discussion of the propagator we already know that

$$(\partial^2 + m^2 - i\epsilon)^{-1}(x) = iD_F(x) \tag{40}$$

Now the assertion that eq.(35) represents the propagator follows directly.

1.3 Fermions, the Dirac equation

Fermions are described by so-called spinor fields usually represented as 4-columns

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_3(x) \end{pmatrix} \tag{41}$$

The 4 degrees of freedom account for the 2 spin degrees of freedom for a fermion and an antifermion. Dirac found the wave equation for spinors by seeking a first order differential equation that would imply the second order Klein-Gordon equation. The most general first order equation may be written as

$$(i\Omega_\mu \partial^\mu - m)\psi(x) = 0 \tag{42}$$

where the Ω_μ's will turn out to be 4×4 matrices, the Dirac gamma matrices, and where m is a constant (times the 4×4 unit matrix) that will turn out to be the mass of the quanta. From the Dirac equation (42) follows a second order equation

$$(i\Omega_\nu \partial^\nu + m)(i\Omega_\mu \partial^\mu - m)\psi = 0 \tag{43}$$

which is easily seen to be the Klein-Gordon equation provided

$$\{\Omega_\mu, \Omega_\nu\} \equiv \Omega_\mu \Omega_\nu + \Omega_\nu \Omega_\mu = 2\eta_{\mu\nu} I \tag{44}$$

with I the 4×4 unit matrix.

Under a Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{45}$$

with (x) and (x') representing the same space-time point in coordinates of different inertial frames, the spinor $\psi(x)$ transforms linearly

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x) \tag{46}$$

with $S(\Lambda)$ a certain 4×4 matrix depending on the Lorentz transformation and satisfying

$$\begin{aligned} S^{-1}(\Lambda)\Omega^\mu S(\Lambda) &= \Lambda^\mu{}_\nu \Omega^\nu \\ \Omega^\mu &\equiv \eta^{\mu\nu}\Omega_\nu \end{aligned} \tag{47}$$

When the transformation is infinitesimal one may obtain (ω infinitesimal)

$$\begin{aligned} \Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \omega^\mu{}_\nu \\ \omega_{\mu\nu} &= -\omega_{\nu\mu} \\ S(\Lambda) &= 1 - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu} \\ \sigma_{\mu\nu} &\equiv \frac{i}{2}[\Omega_\mu, \Omega_\nu] \end{aligned} \tag{48}$$

The reader may verify that a possible realization of the gamma matrices is the set

$$\begin{aligned} &\Omega^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} ; \; \Omega^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\ &\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{49}$$

where of course the σ^i are the Pauli matrices and where I is the 2×2 unit matrix and 0 sometimes represents the 2×2 matrix with 0 entries.

It is the transformation rule eq.(48) that implies that quanta described by these spinors have spin $\frac{1}{2}$. Namely one learns about the spin by studying transformation properties under rotations, and these are a subset of the general Lorentz transformations.

One further defines the adjoint spinor $\overline{\psi}\equiv \psi^\dagger \Omega^0$ which transforms under Lorentz transformations as

$$\overline{\psi}(x) \to \overline{\psi}'(x') = \overline{\psi}(x) S^{-1}(\Lambda) \tag{50}$$

so that it is very easy to build objects transforming in simpler ways under Lorentz transformations, for example

$$\begin{aligned} &\overline{\psi}(x)\psi(x) \quad \text{scalar} \\ &\overline{\psi}(x)\Omega^\mu\psi(x) \quad \text{4-vector} \end{aligned} \tag{51}$$

Parity $(t,\vec{x}) \to (t,-\vec{x})$ is a particular example of a Lorentz transformation $\Lambda = P$ for which one verifies that $S(P) = \Omega^0$. The chirality matrix

$$\begin{aligned} &\Omega^5 = \Omega_5 = i\Omega^0\Omega^1\Omega^2\Omega^3 \\ &(\Omega^5)^\dagger = \Omega^5; \;\; (\Omega^5)^2 = 1; \;\; \{\Omega^5, \Omega^\mu\} = 0 \end{aligned} \tag{52}$$

plays a crucial role in weak interactions; we shall not use it much here. One finds the following transformation properties under Lorentz transformations

$$\begin{aligned} &\overline{\psi}(x)\Omega^5\psi(x) \quad \text{pseudoscalar} \\ &\overline{\psi}(x)\Omega^\mu\Omega^5\psi(x) \quad \text{axial 4-vector} \end{aligned} \tag{53}$$

Suitable lagrangians for free fermions are

$$\mathcal{L} = \overline{\psi}(x)(i\Omega_\mu\partial^\mu - m)\psi(x) \sim -\partial^\mu\overline{\psi}(x)(i\Omega_\mu + m)\psi(x) \tag{54}$$

These two give rise to the same action after partial integration. The first one gives the Dirac equation for $\psi(x)$ by the Euler-Lagrange equation for (a component of) $\overline{\psi}$, whereas the second gives the Dirac equation for $\overline{\psi}$by the EL equation for ψ.

Let us introduce Feynman's *slash* notation. For any 4-vector a_μ we write $\not{a} \equiv a_\mu\Omega^\mu$. Thus the action may be written

$$S_{\text{free Dirac}} = \int d^4x \overline{\psi}(x)(i\not{\partial} - m)\psi(x) \tag{55}$$

Chiral components are defined by

$$\psi = \psi_L + \psi_R = \frac{1}{2}(1-\Omega_5)\psi + \frac{1}{2}(1+\Omega_5)\psi \tag{56}$$

Verify that

$$S = \int d^4x \{\overline{\psi}_L i\not{\partial}\psi_L + \overline{\psi}_R i\not{\partial}\psi_R - m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R\} \tag{57}$$

showing that left and right degrees of freedom separate for $m=0$ in which case the two represent definite helicity $\pm\frac{1}{2}$.

We shall need the plane wave solutions of the Dirac equation. Let us represent these as

$$u_s(p)e^{-ipx}, \;\; v_s(p)e^{+ipx} \tag{58}$$

for positive and negative energy solutions respectively. There are 4 linearly independent solutions in total, labelled by $s = \pm\frac{1}{2}$ indicating spin. Clearly the Klein-Gordon equation demands $p^2 = m^2$ and we take in both cases $p^0 = +\sqrt{\vec{p}^2 + m^2}$. Then the Dirac equation for u and v spinors becomes

$$\begin{aligned} \not{p}u(p) &= mu(p) \\ \not{p}v(p) &= -mv(p) \end{aligned} \tag{59}$$

A convenient Lorentz invariant normalization is

$$\begin{aligned} \overline{u}_s(p)u_{s'}(p) &= 2m\delta_{ss'} \\ \overline{v}_s(p)v_{s'}(p) &= -2m\delta_{ss'} \end{aligned} \tag{60}$$

Having found a complete set of solutions to the Dirac equation we are ready to write an arbitrary Heisenberg quantum field operator satisfying the free Dirac equation as a superposition of these with certain operator coefficients:

$$\psi(x) = \sum_{\vec{p}} \sum_{s} \left\{ b_s(\vec{p})u_s(\vec{p})e^{-ips} + d_s^\dagger(\vec{p})v_s(\vec{p})e^{+ipx} \right\} \tag{61}$$

with $p^0 \equiv +\sqrt{\vec{p}^2 + m^2}$ as usual. So we have introduced fermion annihilation and creation operators $b_s(\vec{p}), b_s^\dagger(\vec{p})$ and anti-fermion annihilation and creation operators $d_s(\vec{p}), d_s^\dagger(\vec{p})$, which we shall take to satisfy the *anti commutation relations*

$$\begin{aligned} \{b_s(\vec{p}), b_{s'}^\dagger(\vec{p}')\} &= \delta_{\vec{p}\vec{p}'}\delta_{ss'} \\ \{d_s(\vec{p}), d_{s'}^\dagger(\vec{p}')\} &= \delta_{\vec{p}\vec{p}'}\delta_{ss'} \\ \{b_s(\vec{p}), b_{s'}(\vec{p}')\} &= 0 = \{b_s^\dagger(\vec{p}), b_{s'}^\dagger(\vec{p}')\} \\ \{d_s(\vec{p}), d_{s'}(\vec{p}')\} &= 0 = \{d_s^\dagger(\vec{p}), d_{s'}^\dagger(\vec{p}')\} \end{aligned} \tag{62}$$

with

$$\{A, B\} \equiv AB + BA \tag{63}$$

These imply Fermi-Dirac statistics and the Pauli principle for the particles. The propagator is constructed entirely like in the bosonic case, either by inverting the wave operator (the Dirac equation) or by considering the two point function (a 4×4 Dirac matrix!)

$$S_F(x-y) = \langle 0|T\{\psi(x)\overline{\psi}(y)\}|0\rangle \tag{64}$$

where now the time ordering operator is defined such that one takes the above order for $x^0 > y^0$ and the opposite order *with an extra minus sign* for Fermi statistics, when $y^0 > x^0$. The result is easily seen to be

$$\begin{aligned} S_F(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ipx}}{\not{p} - m + i\epsilon} \\ &= i\int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} e^{-ipx} \\ &= (\not{p} + m)D_F(x) \end{aligned} \tag{65}$$

One may worry here as well as in the case of the scalar propagator, that the time ordering can be ambiguous. Indeed for x and y separated by a space-like distance this ambiguity occurs. Consistency then requires that the two field operators commute in the bosonic case and anticommute in the fermionic case so that we get the same result whatever the ordering. This makes sense since spatially separated events cannot be causally connected and therefore the quantum field operators should in some sense be independent. One may check that with the commutation relations we have imposed for bosons and for fermions, this vanishing of the (anti) commutator for operators that are spatially separated indeed takes place. One may similarly check that an inconsistency would arise were we to quantize the Dirac quanta by commutators or the scalar quanta by anti commutators. This is the heart of the celebrated spin statistics theorem.

1.4 Free Quantum Electro Dynamics

In a Lorentz invariant framework Maxwell's equations are best expressed in terms of the field strength tensor $F^{\mu\nu} = -F^{\nu\mu}$ built from the electric and magnetic fields $\vec{E}, \vec{B}$as

$$\begin{aligned} F^{0i} &= -E_i \\ F^{lk} &= -\epsilon_{lkj} B_j \end{aligned} \tag{66}$$

($\epsilon_{123} = +1 =$ even permutations $= -$ odd permutations). Also we use the 4-current density built from the charge density ρ and the current density $\vec{j}$ as $j^\mu = (\rho, \vec{j})$. Then the Maxwell equations take the covariant form

$$\begin{aligned} \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= 0 \\ \partial_\mu F^{\mu\nu} &= j^\nu \end{aligned} \tag{67}$$

Charge conservation implies the continuity equation $\partial_\mu j^\mu = 0$. The first set of Maxwell equations (the ones without currents) imply the existence of a 4-vector potential A_μ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{68}$$

The field strength tensor is invariant under the gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi \tag{69}$$

with $\chi(x)$ an arbitrary scalar function. The lagrangian density

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \tag{70}$$

gives rise to the equations of motion $\partial_\mu F^{\mu\nu} = j^\nu$ under variation after A_μ, the other "equation of motion" being treated as an identity satisfied by using the 4-vector potential as the field. From the equation of motion for $F^{\mu\nu}$ follows the equation of motion for A_μ itself:

$$\partial^2 A_\mu - \partial_\mu (\partial^\nu A_\nu) = j_\mu \tag{71}$$

Due to the presence of the second term this is not the Klein-Gordon equation for a massless photon field (for $j_\mu \equiv 0$), except if we also impose the Lorentz-gauge

$$\partial^\nu A_\nu = 0 \tag{72}$$

Because of gauge invariance eq.(69) we cannot construct a meaningful photon propagator immediately by inverting the differential operator acting on the photon field in eq.(71). This is because a given current in fact does not at all uniquely determine the radiation field A_μ, since for any one solution we may perform a gauge transformation eq.(69) on it an obtain a new field containing exactly the same physics.

The solution to this problem consists in choosing a gauge in one way or another. For example we may add a *gauge fixing term* to the Maxwell lagrangian of the form

$$-\frac{1}{2\xi}(\partial^\mu A_\mu)^2 \tag{73}$$

where ξ is an arbitrary parameter labelling a whole family of possible gauges. Evidently if we do that in the path integral, certain things obtained from the path integral will depend on ξ. But one may show that quantum amplitudes involving *gauge invariants* are independent of ξ (or indeed of any way in which we may choose to impose a gauge condition). Including the gauge fixing term, the part of the action bilinear in A_μ may be written (after a partial integration)

$$S_{\text{QED}}^{\text{gauge fixed}} = \frac{1}{2}\int d^4x A_\nu(x)\{\partial^2\eta^{\nu\mu} - \left(1-\frac{1}{\xi}\right)\partial^\nu\partial^\mu\}A_\mu(x) \tag{74}$$

This differential operator may be inverted with the resulting photon propagator

$$D_F^{\mu\nu}(s) = -i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\epsilon}\left(\eta^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right) \tag{75}$$

as is easily verified. Often we shall use the *Feynman gauge* $\xi = 1$

$$D_F^{\mu\nu} = \frac{-i\eta^{\mu\nu}}{k^2+i\epsilon} \tag{76}$$

The quantum field operator may be represented as

$$A^\mu(x) = \sum_{\vec{k}}\sum_\rho\left(\epsilon_\rho^\mu(\vec{k})a^\rho(\vec{k})e^{-ikx} + \epsilon_\rho^{\mu *}(\vec{k})a^{\rho\dagger}(\vec{k})e^{ikx}\right) \tag{77}$$

Here $\epsilon^\mu(\vec{k})$ is a polarization vector related to the 3-vector $\vec{k}$ and further specified by the polarization index ρ. As is well known, physical photons possess 2 polarization states. This may be made manifest by going to a *unitary gauge* such as the *radiation gauge* $A^0 \equiv 0 \equiv \vec{\nabla}\cdot\vec{A}$. In that gauge $\epsilon^0(\vec{k}) = 0 = \vec{k}\cdot\vec{\epsilon}(\vec{k})$ so that we have the expected two polarization vectors perpendicular to the momentum $\vec{k}$. Taking them orthonormal they are seen to satisfy

$$\sum_{i=1}^{2}\epsilon_i^l(\vec{k})\epsilon_i^m(\vec{k}) = \delta^{lm} - \frac{k^l k^m}{\vec{k}^2} \tag{78}$$

The photon creation and annihilation operators pertaining to the 2 polarization states satisfy

$$[a_i(\vec{k}), a_j^\dagger(\vec{k}')] = \delta_{ij}\delta_{\vec{k}\vec{k}'} \tag{79}$$

This gauge leads to Lorentz non-invariant intermediate calculations and it is not obvious that the scheme is at all Lorentz invariant. Therefore one often prefers to use the covariant gauges above, with a formal polarization index ρ running over $0,1,2,3$ and creation and annihilation operators satisfying

$$[a^{\rho\dagger}(\vec{k}), a^\sigma(\vec{k}')] = -\eta^{\rho\sigma}\delta_{\vec{k}\vec{k}'} \tag{80}$$

These agree with the physical ones eq.(79), but involve 2 additional unphysical photons, a longitudinal one $\rho = 3$, and a scalar one $\rho = 0$. The scalar photon has “wrong sign commutation relations” that imply a scalar photon state with “negative norm” or “negative probability”, a non-sensical or “non-unitary” property. However, it turns out to be consistent to carry out calculations where the unphysical photons are produced in the process provided we always sum over the associated probabilities, since the two unphysical probabilities precisely cancel each other. In this case the polarization tensor eq.(78) gets replaced by

$$\sum_{\rho=0}^{3} \epsilon^{\mu}_{\rho}(\vec{k})\epsilon^{\nu\rho}(\vec{k}) = \eta^{\mu\nu} - (1-\xi)\frac{k^{\mu}k^{\nu}}{k^2} \tag{81}$$

in the "ξ" gauge introduced above. One notices the occurence of the same tensor as in the photon propagator. It is possible (but non trivial) to prove that any gauge, whether unitary or non unitary gives rise to identical predictions for gauge invariant objects.

2 LECTURE 2. CROSS SECTIONS, FEYNMAN RULES. NON-ABELIAN GAUGE THEORY (QCD)

2.1 S-matrix and cross section

Scattering experiments are described by means of the S-matrix and the T-matrix, the two being related by

$$\langle f|S|i\rangle = \langle f|i\rangle + i(2\pi)^4\delta^4(P_f - P_i)\langle f|T|i\rangle \tag{82}$$

where i, f denote initial and final states with total 4-momentum P_i and P_f. We shall learn how to calculate the T-matrix (often also called T_{fi} or $\mathcal{M}_{fi}$) by means of Feynman rules. It is related to the cross section in a process of the form

$$a + b \to c_1(p_1) + ... + c_n(p_n) \tag{83}$$

by

$$\begin{aligned} d\sigma_{fi} &= \frac{1}{2\sqrt{\lambda(s, m_a^2, m_b^2)}}|T_{fi}|^2 \\ &\times\ (2\pi)^4\delta^4(p_a + p_b - \sum_i p_i)\prod_{j=1}^{n}\frac{d^3p_j}{(2\pi)^3 2E_j} \\ E_j &\equiv \sqrt{\vec{p}_j^{\,2} + m_j^2} \\ \lambda(x,y,z) &= x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \\ 2\sqrt{\lambda(s, m_a^2, m_b^2)} &= 4m_b|\vec{p}_1^{\,\mathrm{lab}}| = ... \end{aligned} \tag{84}$$

We use state vector normalizations derived from the commutation relations between creation and annihilation operators in lecture 1; thus for a boson

$$\begin{aligned} \langle\vec{p}\,'|\vec{p}\rangle &= \langle 0|a(\vec{p}\,')a^{\dagger}(\vec{p})|0\rangle = \delta_{\vec{p}\,'\vec{p}} = 2p^0(2\pi)^3\delta^3(\vec{p}\,' - \vec{p}) \\ \langle\vec{p}|\vec{p}\rangle &= 2p^0(2\pi)^3\delta^3(0) = 2p^0 V \end{aligned} \tag{85}$$

where V is a quantization volume (cf. $(2\pi)^3\delta(\vec{p}) = \int d^3x e^{i\vec{p}\cdot\vec{x}} \Rightarrow (2\pi)^3\delta^3(0) = V$).

The differential scattering cross section for the process

$$a + b \to c_1(p_1) + c_2(p_2)$$

takes the following form in the centre of mass frame

$$\begin{aligned} \frac{d\sigma}{d\Omega_{c.m.}} &= \frac{p_f}{p_i}\left|\frac{T_{fi}}{8\pi\sqrt{s}}\right|^2 \\ s &= (p_a + p_b)^2 = (\text{total c.m. energy})^2 \\ p_i &= |\vec{p}_a| = |\vec{p}_b| \\ p_f &= |\vec{p}_1| = |\vec{p}_2| \end{aligned} \tag{86}$$

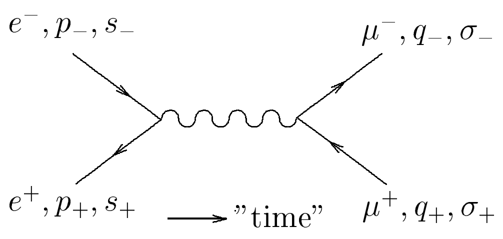


Fig. 1: Feynman diagram for the process $e^- e^+ \to \mu^- \mu^+$ to lowest order in the fine structure constant.

2.2 $e^- e^+ \to \mu^- \mu^+$

In this section we want to "derive" the QED Feynman rules for this very famous monitoring process. This will serve as an illustration of how the rules are derived in general. We only consider the lowest order in the fine structure constant $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$. The well known Feynman diagram for the process is given by fig.1. Time is flowing from left to right. Fermions are represented by solid lines with arrows in the time direction for fermions and against the time direction for antifermions. The wavy line represents a photon propagator. One builds T_{fi} as a product of several factors: one for each fermion line and one for the photon propagator. Each of these is a complex number. The numbers for fermion lines is built as a matrix product with the structure (adjoint spinor)$\cdot$ (4×4 matrix (matrices))$\cdot$(spinor). Remembering that adjoint spinors are row matrices and spinors are column matrices, this indeed gives a number. One picks up the different pieces by walking *against* the fermion arrow. External fermions give u-spinors (or adjoints); external antifermions give v-spinors (or adjoints). Vertices in QED give a factor $-ie\Omega^\mu$ and the photon propagator we have already met. It has two Lorentz-indices which must be used to glue together whatever Lorentz-indices come on either sides of it by way of vertices. Here we shall use the Feynman gauge. With the notation of the figure we get

$$\begin{aligned} iT_{fi} &= \overline{v}_{s_+}(p_+)(-ie\Omega_\mu)u_{s_-}(p_-) \quad \text{(electron line)} \\ &\cdot\ \overline{u}_{\sigma_-}(q_-)(-ie\Omega_\nu)v_{\sigma_+}(q_+) \quad \text{(muon line)} \\ &\cdot\ \frac{-i\eta^{\mu\nu}}{k^2+i\epsilon} \quad \text{(photon propagator)} \\ k^\mu &= p_-^\mu + p_+^\mu = q_-^\mu + q_+^\mu \\ s &= k^2 \end{aligned} \tag{87}$$

One uses 4-momentum conservation in all vertices. Thus

$$iT_{fi} = i\frac{e^2}{s}\,(\overline{v}(p_+)\Omega_\mu u(p_-))(\overline{u}(q_-)\Omega^\mu v(q_+)) \tag{88}$$

Notice that the $i\epsilon$ is irrelevant here since $s \gg 0$ for a physical process. Also notice that the spinors satisfy different Dirac equations

$$\begin{aligned} \not{p}_- u(p_-) &= m_e u(p_-) \\ \not{q}_- u(q_-) &= m_\mu u(q_-) \end{aligned} \tag{89}$$

etc. Fig.2 summarizes the Feynman rules of QED. In addition there are some further complications

- *extra minus signs* occur with fermions whenever
  - fermion lines cross (pick a definite vertical ordering of external lines)
  - an antifermion line passes all the way from initial to final state
  - fermion loops are present (there is a factor -1 for each loop)
- *weight factors* different from 1 may occur for diagrams with loops. We shall give examples in lecture 3.

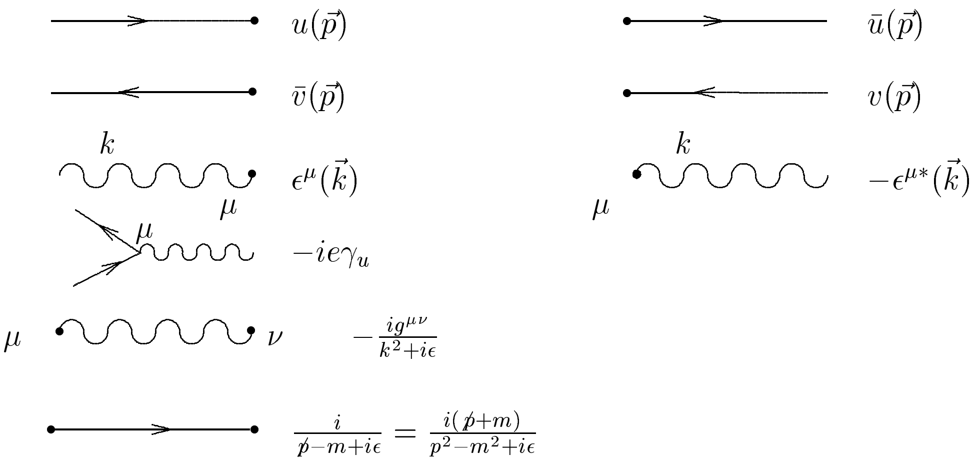


Fig. 2: Feynman rules of QED.

2.2.1 Dyson's formula

Dyson's formula for the S-matrix is

$$S = T\exp\{i\int d^4x\mathcal{L}_I(\phi,\partial_\mu\phi)\} \tag{90}$$

where the precise meaning of the Time-ordering sign will be presented below and where the *interaction* lagrangian density $\mathcal{L}_I$ (for notational reasons only) is taken to depend on just one scalar field. The field operators are in the *interaction picture* where they have a free field time dependence even when non trivial interactions are present, so that our description in lecture 1 in terms of creation and annihilation operators can still be used. This formula is often derived in elementary quantum mechanics treatments of time-dependent perturbation theory. Here for completeness we briefly indicate how it is derived leaving some details to the reader. We do this in the traditional operator formulation. Later we shall briefly indicate how to do it with path integrals.

We split the *hamiltonian* in a free part and and interaction part

$$H = H_0 + H_I \tag{91}$$

In the Schrödinger picture a time dependent state satisfies the Schrödinger equation

$$i\partial_t|t\rangle_S = (H_0 + H_I)|t\rangle_S \tag{92}$$

Define the *interaction picture* by

$$|t\rangle_I \equiv e^{itH_0}|t\rangle_S \tag{93}$$

It satisfies

$$\begin{aligned} i\partial_t|t\rangle_I &= H_I(t)|t\rangle_I \\ H_I(t) &\equiv e^{itH_0}H_Ie^{-itH_0} \end{aligned} \tag{94}$$

Introduce the time evolution operator by

$$|t\rangle_I = U(t,t_0)|t_0\rangle_I \tag{95}$$

It satisfies

$$i\partial_t U(t,t_0) = H_I(t)U(t,t_0) \tag{96}$$

Together with the boundary condition $U(t_0, t_0) = 1$, this implies the integral equation

$$U(t, t_0) = 1 - i \int_{t_0}^{t} dt' H_I(t') U(t', t_0) \tag{97}$$

which we may solve by iteration to obtain

$$U(t, t_0) = I - i \int_{t_0}^{t} dt' H_I(t') + (-i)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_I(t') H_I(t'') + ... \tag{98}$$

Finally the S-matrix $U(-\infty, +\infty)$ is obtained as

$$\begin{aligned} U(-\infty, \infty) \;=\; & \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \int_{-\infty}^{\infty} d^4x_2 ... \int_{-\infty}^{\infty} d^4x_n \\ & T\{\mathcal{H}_I(x_1) ... \mathcal{H}_I(x_n)\} \end{aligned} \tag{99}$$

For simple theories without derivative interactions $\mathcal{H}_I = -\mathcal{L}_I$. Thus we obtain Dyson's formula eq.(90).

2.2.2 Derivation of the amplitude for $e^- e^+ \to \mu^- \mu^+$

For this process we use the free lagrangian

$$\mathcal{L}_{\text{free}} = \overline{\psi}_e (i \not\partial - m_e) \psi_e + \overline{\psi}_\mu (i \not\partial - m_\mu) \psi_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{100}$$

(where we have left out gauge fixing for simplicity). The interaction part we take as

$$\mathcal{L}_I = -j_\mu A^\mu = -\left(e \overline{\psi}_e \Omega_\mu \psi_e + \overline{\psi}_\mu \Omega_\mu \psi_\mu \right) A^\mu \tag{101}$$

(We hope that there is no confusion coming from the fact that μ is used both to denote a Lorentz index and to denote the name of the muon particle). The precise form of the elctromagnetic current used here is obtained by replacing ∂_μ in the free lagrangian for electrons and muons by $\partial_\mu - ieA_\mu$. We shall see towards the end of this lecture how that follows in a gauge theory. For now we merely remark that this current is conserved by virtue of the Dirac equation as is easily verified.

Now remember that

$$\psi(x) = \sum_{\vec{p}, s} \left(u_s(\vec{p}) b_s(\vec{p}) e^{-ipx} + v_s(\vec{p}) d_s^\dagger(\vec{p}) e^{ipx} \right) \tag{102}$$

annihilates fermions and creates antifermions, whereas

$$\overline{\psi}(x) = \sum_{\vec{p}, s} \left(\overline{u}_s(\vec{p}) b_s^\dagger(\vec{p}) e^{ipx} + \overline{v}_s(\vec{p}) d_s(\vec{p}) e^{-ipx} \right) \tag{103}$$

creates fermions and annihilates antifermions. Thus to obtain a non-vanishing contribution to

$$\langle \mu^-(q_-) \mu^+(q_+) | S | e^-(p_-) e^+(p_+) \rangle$$

we must annihilate an electron, annihilate an anti-electron, create a muon and create an anti-muon. Hence we must expand the S-matrix according to Dyson's formula to at least 2nd order. This gives the lowest order non vanishing contribution. The result becomes (leaving out terms that give zero)

$$\begin{aligned} \langle \mu^-(q_-) \mu^+(q_+) | S | e^-(p_-) e^+(p_+) \rangle \;=\; & -\frac{2}{2} e^2 \int d^4x d^4y \langle \mu^-(q_-) \mu^+(q_+) | \\ & T\{\overline{\psi}_\mu(y) \not{A}(y) \psi_\mu(y) \overline{\psi}_e(x) \not{A}(x) \psi_e(x) \\ & | e^-(p_-) e^+(p_+) \rangle \end{aligned} \tag{104}$$

Now work out

$$\langle 0|\psi_e(x)|e^-(p_-,s_-)\rangle = u_{s_-}(\vec{p}_-)e^{-ip_-x} \tag{105}$$

etc. Also we have seen in lecture 1 that $\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle$ is the photon propagator. The various "plane wave factors" will produce 4-momentum conservation at the two vertices ((x) and (y)) after integration over x and y. This completes the derivation of the Feynman amplitude for $e^-e^+ \to \mu^-\mu^+$ eq.(88) from Dyson's formula eq.(90).

2.2.3 The cross section for $e^-e^+ \to \mu^-\mu^+$

We now want to work out the cross section according to eq.(86) based on the amplitude eq.(88). Thus we must learn how to square amplitudes. Also since most detectors trigger on particles no matter what their spin is, we must sum over final spins to compare with results from such detectors, and since most accelerators give unpolarized beams we must perform an average over cross sections for the various possible initial spins. Thus we must sum over all spins and divide by the number of initial spin states, here $2\cdot 2 = 4$. The absolute square of the amplitude is the amplitude times it's complex conjugate. The latter is best evaluated as the *hermitean conjugate* of the matrix products (that result in 1×1 matrices). Thus

$$\begin{aligned}
\sum_{\text{spin}} |T_{fi}|^2 &= \sum_{\text{spin}} T_{fi}\cdot T_{fi}^* \\
&= \frac{e^4}{s^2}\sum_{\text{spin}} (E_\mu M^\mu)(E_\nu M^\nu)^* \\
M^\mu &= \overline{u}(\vec{q}_-)\varOmega^\mu v(\vec{q}_+) \\
E_\mu &= \overline{v}(\vec{p}_+)\varOmega_\mu u(\vec{p}_-) \\
\sum_{\text{spin}} E_\mu E_\nu^* \equiv E_{\mu\nu} &= \sum_{s_+,s_-} \overline{v}_{s_+}(\vec{p}_+)\varOmega_\mu u_{s_-}(\vec{p}_-)u_{s_-}^\dagger \varOmega_\nu^\dagger \varOmega_0^\dagger v_{s_+}(\vec{p}_+)
\end{aligned} \tag{106}$$

where we used $\overline{v} = v^\dagger \varOmega_0$. Now check that

$$\varOmega_0^\dagger = \varOmega_0 = \varOmega^0, \quad \varOmega_0^2 = I, \quad \varOmega_0\varOmega_\mu^\dagger\varOmega_0 = \varOmega_\mu \tag{107}$$

Then

$$\begin{aligned}
E_{\mu\nu} &= \sum_{s_+,s_-} \overline{v}_{s_+}(\vec{p}_+)\varOmega_\mu u_{s_-}(\vec{p}_-)\overline{u}_{s_-}(\vec{p}_-)\varOmega_\nu v_{s_+}(\vec{p}_+) \\
&= \sum_{s_+,s_-} Tr\{\left(v_{s_+}(\vec{p}_+)\overline{v}_{s_+}(\vec{p}_+)\right)\varOmega_\mu\left(u_{s_-}(\vec{p}_-)\overline{u}_{s_-}(\vec{p}_-)\right)\varOmega_\nu\}
\end{aligned} \tag{108}$$

Here we used that for a row matrix $\overline{V}$ and a column matrix U

$$\overline{V}U = Tr\{U\overline{V}\} \tag{109}$$

We now use the spin projection property

$$\begin{aligned}
\sum_s u_s(\vec{p})\overline{u}_s(\vec{p}) &= \not{p} + m \\
\sum_s v_s(\vec{p})\overline{v}_s(\vec{p}) &= \not{p} - m
\end{aligned} \tag{110}$$

These identities between 4×4 matrices are proven by verifying that the left hand sides and the right hand sides have identical actions on the 4 linearly independent spinors $u_{s'}(\vec{p}), v_{s'}(\vec{p})$. That in turn is easy to show using the normalization relations and the Dirac equations for these spinors given in lecture 1. Hence

$$E_{\mu\nu} = Tr\{(\not{p}_+ - m_e)\Omega_\mu(\not{p}_- + m_e)\Omega_\nu\} \tag{111}$$

We now provide a small toolbox for doing traces of gamma matrices

$$\begin{aligned}
Tr(AB) &= Tr(BA) \Rightarrow Tr(AB...CD) = Tr(DAB...C) \\
Tr(\Omega_\mu) &= 0 = Tr(\text{odd number of } \Omega'_\mu s) \\
(\text{Proof} &: Tr(\Omega_{\mu_1}...\Omega_{\mu_n}) = \text{tr}(\Omega_5^2\Omega_{\mu_1}...\Omega_{\mu_n}) = Tr(\Omega_5\Omega_{\mu_1}...\Omega_{\mu_n}\Omega_5) \\
&= -Tr(\Omega_5\Omega_{\mu_1}...\Omega_5\Omega_{\mu_n}) = ... = (-)^n Tr(\Omega_5^2\Omega_{\mu_1}...\Omega_{\mu_n})) \\
Tr\{\Omega_\mu\Omega_\nu\} &= 4\eta_{\mu\nu} \\
Tr\{\Omega_\mu\Omega_\nu\Omega_\rho\Omega_\sigma\} &= 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\}
\end{aligned} \tag{112}$$

The last two are obtained using the anti commutation relations for the gamma matrices (and some work for the last one).

With the help of these rules we rather easily obtain

$$\begin{aligned}
E_{\mu\nu} &= 4(p_{-\mu}k_\nu + p_{-\nu}k_\mu - 2p_{-\mu}p_{-\nu} - \eta_{\mu\nu}k\cdot p_-) \\
M^{\mu\nu} &= 4(q_-^\mu k^\nu + q_-^\nu k^\mu - 2q_-^\mu q_-^\nu - \eta^{\mu\nu}k\cdot q_-)
\end{aligned} \tag{113}$$

($k = p_- + p_+$). Let us choose the frame so that

$$\begin{aligned}
p_-^\mu &= E(1,0,0,\beta_e) \\
q_-^\mu &= E(1,0,\beta_\mu\sin\theta,\beta_\mu\cos\theta) \\
4E^2 &= s
\end{aligned} \tag{114}$$

Then

$$\begin{aligned}
E_{\mu\nu}M^{\mu\nu} &= 4s^2\{1 + \beta_e^2\beta_\mu^2\cos^2\theta + \frac{4}{s}(m_e^2 + m_\mu^2)\} \\
\left(\frac{d\sigma}{d\Omega}\right)_{c.m.} &= \frac{\alpha^2}{4s}\frac{\beta_m u}{\beta_e}\{1 + \beta_e^2\beta_\mu^2\cos^2\theta + \frac{4}{s}(m_e^2 + m_\mu^2)\} \\
\sigma(e^-e^+ \to \mu^-\mu^+) &= \frac{\alpha^2\pi}{s}\frac{\beta_\mu}{\beta_e}\left(1 + \frac{1}{3}\beta_e^2\beta_\mu^2 + \frac{4}{s}(m_e^2 + m_\mu^2)\right) \\
&\stackrel{s\to\infty}{\to} \frac{4}{3}\pi\frac{\alpha^2}{s} = \sigma_{pt.} = \frac{87\text{nb}}{s/(\text{GeV})^2}
\end{aligned} \tag{115}$$

2.3 Feynman rules from path integrals

We have already indicated the relationship between the operator formulation and the path integral formulation of quantum field theory. It is essentially the statement that Greens functions are given by the following two expressions (for a scalar field theory for simplicity)

$$\begin{aligned}
G(x_1,...,x_n;y_1,...,y_m) &= \langle 0|T\{\phi(x_1)...\phi(x_n)\phi(y_1)...\phi(y_m)\}|0\rangle \\
&= N^{-1}\int \mathcal{D}\phi e^{iS}\phi(x_1)...\phi(x_n)\phi(y_1)...\phi(y_m)
\end{aligned} \tag{116}$$

These Greens functions are related to S-matrix elements as follows: first take the Fourier transform $G(p_1,...,p_n;k_1,...,k_m)$ say pertaining to n incoming and m outgoing particles. Here in general the 4-momenta need not be on the mass shells $p_i^2 = m_i^2$, but in the limit where they do the Greens function develops poles in p_i^2 and k_j^2. The *residue* of all poles is the S-matrix element (up to a certain wave function renormalization).

It turns out to be simplest to first develop Feynman rules for these Greens functions. As in the case of the free theory we introduce the generating function of Greens functions,

$$\begin{aligned} Z(J) &= \int \mathcal{D}\phi e^{iS(\phi)+i\int d^4xJ(x)\phi(x)} \\ &= \int \mathcal{D}\phi e^{iS} \sum_n \frac{i^n}{n!} \int d^4x_1...d^4x_n J(x_1)...J(x_n)\phi(x_1)...\phi(x_n) \\ &= \sum_n \frac{i^n}{n!} \int d^4x_1...d^4x_n J(x_1)...J(x_n) G(x_1,...,x_n) \end{aligned} \tag{117}$$

so that

$$G(x_1,...,x_n) = \frac{\delta}{iJ(x_1)} \cdots \frac{\delta}{iJ(x_n)} Z(J)|_{J\equiv 0} \tag{118}$$

For the free theory we found

$$Z_0(J) = \int \mathcal{D}\phi e^{iS_0(\phi)+iJ\cdot\phi} = Z(0)\exp\{-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)\} \tag{119}$$

($J\cdot\phi \equiv \int d^4xJ(x)\phi(x)$) and for any function $f(\phi)$ of the fields we have

$$\int \mathcal{D}\phi e^{iS_0(\phi)} f(\phi) = f(\frac{\delta}{i\delta J}) \int \mathcal{D}\phi e^{iS_0+J\cdot\phi}|_{J\equiv 0} = f(\frac{\delta}{iJ}) Z_0(J)|_{J\equiv 0} \tag{120}$$

In particular we get for the full generating function including interactions described by $S_I(\phi)$

$$\begin{aligned} Z(J) &= \int \mathcal{D}\phi e^{iS_0(\phi)+iS_I(\phi)+iJ\cdot\phi} = \int \mathcal{D}\phi e^{iS_0(\phi)+iJ\cdot\phi} e^{iS_I(\phi)} \\ &= e^{iS_I(\frac{\delta}{i\delta J})} \int \mathcal{D}\phi e^{iS_0(\phi)+iJ\cdot\phi} \end{aligned} \tag{121}$$

Thus

$$Z(J) = e^{iS_I(\frac{\delta}{iJ})} e^{-\frac{1}{2}\int d^4xd^4yJ(x)D_F(x-y)J(y)} Z_0(0) \tag{122}$$

This may be viewed as a path integral version of Dyson's formula eq.(90). It was derived without assuming that there be no derivative interactions. In other word it will be applicable to non-abelian gauge theories. It provides the most efficient starting point for proving the Feynman rules.

For a general theory based on several bosonic fields the above treatment trivially generalizes. When fermions are present there is a non trivial new point which occurs in the path integral formalism. We only make a very brief mention of that. The point is that in the path integral fermion fields have to be treated as so-called Grassmann valued fields. To motivate the notion, remember the commutation relations between bosonic creation and annihilation operators

$$[a(\vec{p}), a^\dagger(\vec{p}')] = \hbar\delta_{\vec{p}\vec{p}'} \tag{123}$$

where we have explicitly introduced Planck's constant. In the classical limit $\hbar \to 0$ the commutator vanishes implying that the operators may be replaced by ordinary complex numbers. Hence we use ordinary complex valued classical fields in the path integral. For fermions, however, we have the *anticommutation* relations

$$\{b_s(\vec{p}), b^\dagger_{s'}(\vec{p}')\} = \hbar\delta_{ss'}\delta_{\vec{p}\vec{p}'} \tag{124}$$

so in the classical limit $\hbar \to 0$, $b^\dagger_s(\vec{p}), b_{s'}(\vec{p}')$ *anticommute* and cannot be represented by ordinary (complex) numbers. Instead they must be represented by Grassmann numbers which indeed precisely anticommute. Also the Grassmann spinor fields $\psi(x), \overline{\psi}(x)$ must be coupled to Grassmann valued (spinor) currents $\eta(x), \overline{\eta}(x)$:

$$\overline{\eta} \cdot \psi = \int d^4x \overline{\eta}(x)\psi(x); \;\; \overline{\psi} \cdot \eta = \int d^4x \overline{\psi}(x)\eta(x) \tag{125}$$

and

$$\frac{\delta}{\delta \overline{\eta}_\alpha(y)} \overline{\eta} \cdot \psi = \psi_\alpha(y); \;\; \frac{\delta}{\delta \eta_\alpha(y)} \overline{\psi} \cdot \eta = -\overline{\psi}_\alpha(y) \tag{126}$$

where the last minus sign arises because Grassmann derivatives are taken to anticommute with Grassmann numbers.

A general function of a single Grassmann number θ may be expanded in a Taylor series, but since $\theta^2 = 0$ (indeed $\theta\theta = -\theta\theta$ since θ anticommutes with itself), the most general function is just a linear function! And it suffices to understand how to carry out the two Grassmann integrals

$$\int d\theta \equiv 0, \;\;\; \int d\theta \theta \equiv 1 \tag{127}$$

It may be shown that these definitions provide a path integral quantum mechanics that agrees with the operator quantum mechanics for fermions. But we shall go into no further details here.

Based on eq.(122) it is possible to derive an efficient rule for obtaining the vertex part of Feynman diagrams corresponding to any interaction term, however complicated. For each field entering this term in the interaction there will be a corresponding leg in the vertex. The vertex is obtained by (1) taking functional derivatives with respect to all such fields of the corresponding iS_I; (2) doing a Fourier transform of the result. To illustrate let us show how the rule works on QED. Thus we consider

$$\begin{aligned} &\frac{\delta}{\delta A_\mu(x_3)} \frac{\delta}{\delta \psi_\alpha(x_2)} \frac{\delta}{\delta \overline{\psi}_\beta(x_1)} \left(-ie \int d^4x \overline{\psi}(x) \Omega^\nu A_\nu(x) \psi(x) \right) \\ &= -ie \int d^4x \delta^4(x-x_1)\delta^4(x-x_2)\delta^4(x-x_3)\Omega^\mu_{\beta\alpha} \end{aligned} \tag{128}$$

Taking the Fourier transform of that, i.e. acting with

$$\int d^4x_1 d^4x_2 d^4x_3 e^{-i(p_1x_1 - p_2x_2 + kx_3)}$$

(signs on momenta depend on whether they are taken to flow into or out of the vertex) finally gives the vertex

$$-ie\Omega^\mu_{\beta\alpha}(2\pi)^4\delta^4(p_1 - p_2 + k) \tag{129}$$

the vertex of QED already given, including the rule that $4-$momentum is conserved in the vertex.

We leave it as an exercise to obtain the vertex of $\lambda\phi^4$ theory from the interaction term

$$-i\frac{\lambda}{4!}\int d^4x \phi^4(x)$$

as

$$-i\lambda(2\pi)^4\delta^4(p_1 + p_2 + p_3 + p_4) \tag{130}$$

Finally we give one first example concerning *weight factors* of Feynman diagrams: Tree diagrams have weight factors $\equiv 1$. Consider as an example the ϕ^4 tree-diagram in fig.3. There are 2 vertices, so we must expand Dyson's formula to second order giving us

$$\frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2$$

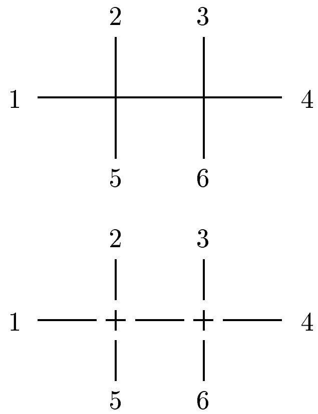


Fig. 3: Example of tree diagram for 6 external scalars. In the second figure, contractions yet have to be performed.

to start with. Beginning with the 6 external lines and the 2 vertices of fig. 3 we must work out in how many ways we can connect lines and vertices while still getting diagrams topologically equivalent with the original one, fig. 3. We see that line 1 may be connected to each of the 8 legs from the 2 vertices, giving a factor of 8, then line 2 must be connected to one of the remaining 3 legs from the same vertex, giving another factor of 3. Continuing this way it is easy to count the total number as

$$8 \cdot 3 \cdot 2 \cdot 4 \cdot 3 = 2 \cdot 4! \cdot 4!$$

so that indeed the weight factor is 1.

2.4 The construction of the lagrangian of (Non-)abelian gauge Theory

For definiteness, let us think mostly of QCD, but the algorithm applies equally to electroweak theory and grand unified models. Thus we consider a theory based on quark fields carrying a colour index $i = 1, 2, 3$. The requirement of a non-abelian gauge theory is that all the physics, including the lagrangian should be invariant under the *gauge transformation*

$$\begin{aligned} q_i(x) &\to q'_i(x) = \mathcal{U}_{ij}(x) q_j(x) \ \text{ or } \ q \to \mathcal{U} q \\ \overline{q}_i(x) &\to \overline{q}'_i(x) = \overline{q}_j(x) \mathcal{U}^\dagger_{ji}(x) \ \text{ or } \ \overline{q} \to \overline{q}\,\mathcal{U}^\dagger \end{aligned} \tag{131}$$

with sums over repeated colour indices understood. The Dirac spinor indices have not been explicitly denoted. The expressions to the right are matrix expressions treating q and $\overline{q}$ as a colour-column and a colour-row respectively. Here we shall take $\mathcal{U}$ to belong the the group $SU(3)$ the group isomorphic to the group of 3×3 matrices with determinant 1. But the construction applies just as well to any simple group, and with minor modifications to non-simple groups as well.

The pertinent point about a gauge transformation is that it is *local* i.e. that $\mathcal{U}(x)$ depends on the space time point x. For constant group elements independent of x one speaks of a global transformation and a corresponding global symmetry. Evidently the free lagrangian of quarks (one flavour only, the case of several flavours is given by a trivial sum over such terms)

$$\mathcal{L}_0 = \overline{q}_i(x)(i \ \partial\!\!\!/ - m) q_i(x) \tag{132}$$

(sum over colour index i implied) is invariant under the global transformation, but due to the presence of ∂_μ in $\partial\!\!\!/$ it cannot be invariant under local gauge transformations. We can repair this by replacing the derivative ∂_μ by a “gauge covariant derivative”

$$\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu(x) \tag{133}$$

where $\mathcal{A}_\mu(x)$ is a new matrix valued gauge field to be identified with the gluon field, and where it is understood that the term ∂_μ is multiplied by the relevant unit matrix: $(\mathcal{D}_\mu)_{ij} = \delta_{ij}\partial_\mu + (\mathcal{A}_\mu)_{ij}$. Now the repaired lagrangian

$$\mathcal{L}_{qqg} = \overline{q}(i\not{\mathcal{D}} + m)q(x) \tag{134}$$

will be gauge invariant provided the covariant derivative transforms "covariantly" that is

$$\mathcal{D}_\mu \to \mathcal{U}\mathcal{D}_\mu\mathcal{U}^\dagger \tag{135}$$

locally. This implies a non trivial transformation of the gauge (matrix) field:

$$\begin{aligned} \mathcal{D}'_\mu &= \partial_\mu + \mathcal{A}'_\mu(x) = \mathcal{U}(x)\{\partial_\mu + \mathcal{A}_\mu(x)\}\mathcal{U}^\dagger(x) \to \\ \mathcal{A}'_\mu(x) &= \mathcal{U}(x)\mathcal{A}_\mu(x)\mathcal{U}^\dagger(x) + \mathcal{U}(x)\partial_\mu\mathcal{U}^\dagger(x) \end{aligned} \tag{136}$$

In the special case of QED we use the abelian group, $U(1)$ consisting of the set of unimodular numbers $\{\mathcal{U} = e^{i\chi(x)}\}$ with χ real. Then $\mathcal{U}$ and $\mathcal{A}_\mu$ are "1×1" matrices, i.e. just numbers and they commute with each other. The gauge transformation for $\mathcal{A}_\mu$ then becomes

$$\mathcal{A}'_\mu(x) = \mathcal{A}_\mu(x) + i\partial_\mu\chi(x) \tag{137}$$

familiar from electrodynamics. Notice that both ∂_μ and (therefore) $\mathcal{A}_\mu(x)$ are antihermitean (remember $p_\mu \sim i\partial_\mu$). Also as we have just seen, an $\mathcal{A}_\mu$ proportional to the unit matrix describes a QED like theory. Hence in our present study of a non-abelian gauge theory we will take $\mathcal{A}_\mu$ to be a linear combination of antihermitean matrices with trace equal to zero, so as to exclude the unit matrix. These are just the matrices of the Lie algebra of $SU(3)$: the linear space of matrices $\mathcal{T}$ such that $\mathcal{U} = 1 + \epsilon\mathcal{T}$ is in $SU(3)$ for $\epsilon \to 0$.

Exercise: show that a $\mathcal{U}$-matrix close to the identity $\mathcal{U} = 1 + i\epsilon\mathcal{T}$ is a matrix in $SU(N)$ for $\epsilon \to 0$ if and only if $\mathcal{T}$ is hermitean and has trace $= 0$.

It is now easy to count that the number of such matrices in $SU(N)$ is $N^2 - 1$. Let us take as a basis of the Lie algebra

$$\{\mathcal{T}^a, \ \ a = 1, ..., N^2 - 1\}$$

Exercise: Argue that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are in the Lie algebra, then $g_1 = e^{i\epsilon_1\mathcal{T}_1}$ and $g_2 = e^{i\epsilon_2\mathcal{T}_2}$ are in the group. Expand the group element $g_1 g_2 g_1^{-1} g_2^{-1}$ to second order in ϵ_1 and ϵ_2 and thereby show that

$$[\mathcal{T}_1, \mathcal{T}_2] \in \text{ Lie algebra} \tag{138}$$

From eq.(138) follows that there exist structure coefficients so that

$$[\mathcal{T}^a, \mathcal{T}^b] = if^{abc}\mathcal{T}^c \tag{139}$$

sum over c implied. It is possible to show that we may always choose linear combinations so that

$$tr(\mathcal{T}^a\mathcal{T}^b) = \frac{1}{2}\delta^{ab} \tag{140}$$

in which case f^{abc} is totally antisymmetric in a, b, c.

Having obtained the "quark-quark-gluon" piece

$$\begin{aligned} \mathcal{L}_{qqg} &= \overline{q}(i\not{\mathcal{D}} - m)q \\ &= \overline{q}(i\not{\partial} - m)q + i\overline{q}\not{\mathcal{A}}q \end{aligned} \tag{141}$$

containing the free quark part plus the interaction part, we ask the question: can there be other Lorentz-invariant and gauge invariant terms that can be added to the lagrangian, in particular can there be terms describing the kinematics and dynamics of the gluon field $\mathcal{A}_\mu$? As is well known the answer is yes, and if we furthermore restrict ourselves to renormalizable terms there is essentially just one possibility (in perturbation theory). To find it we first construct the (matrix valued) field strength tensor

$$\begin{aligned}\mathcal{F}_{\mu\nu}(x) &\equiv [\mathcal{D}_\mu, \mathcal{D}_\nu] = [\partial_\mu + \mathcal{A}_\mu(x), \partial_\nu + \mathcal{A}_\nu(x)] \\ &= \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) + [\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)]\end{aligned} \tag{142}$$

where the last expression demonstrates that despite the construction this is a field, not a differential operator (the proof is by letting the definition act on some suitable field, and then using the chain rule of differentiation). We notice the special case of QED where $\mathcal{A}_\mu(x)$ etc is not a matrix, but just a number. In that case the last commutator in $\mathcal{F}_{\mu\nu}(x)$ vanishes leaving the familiar QED expression. From the definition of $\mathcal{F}_{\mu\nu}(x)$ follows at once that it transforms covariantly under gauge transformations

$$\mathcal{F}'_{\mu\nu}(x) = \mathcal{U}(x)\mathcal{F}_{\mu\nu}(x)\mathcal{U}^\dagger(x) \tag{143}$$

in particular that it is gauge *invariant* in the abelian or QED case. The dimensions of the gauge fields are as follows

$$\begin{aligned}\dim[\partial_\mu] &= \dim[\mathcal{A}_\mu] = L^{-1} \\ \dim[\mathcal{F}_{\mu\nu}] &= L^{-2}\end{aligned} \tag{144}$$

Hence we may write down a possible new Lorentz-invariant and gauge invariant term in the lagrangian

$$\mathcal{L}_{YM} = \frac{1}{2g^2} Tr\{\mathcal{F}_{\mu\nu}(x)\mathcal{F}^{\mu\nu}(x)\} \tag{145}$$

This term is clearly Lorentz invariant. The gauge invariance follows (easy exercise) from the covariant transformation property of the field strength tensor and from the cyclic property of the trace. The dimension of the lagrangian is L^{-4} since the action is dimensionless (in units where $\hbar = 1$). It follows that the gauge theory "coupling constant" g is dimensionless. This property turns out to be linked to renormalizability. Clearly we could add new terms to the lagrangian which would be Lorentz-invariant and gauge invariant, simply by taking powers of the term in $\mathcal{L}_{YM}$. But for dimensional reasons such terms would have to be multiplied by dimensionful coupling constants, and therefore turn out not to be renormalizable. We shall not prove that, but we shall discuss aspects of renormalization theory in lecture 3.

We are now ready to write down the full QCD lagrangian in the matrix notation

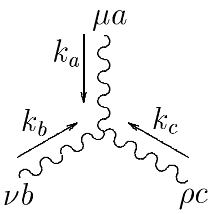
$$\mathcal{L}_{QCD} = \sum_f \overline{q}^f (i \not{\mathcal{D}} - m_f) q^f + \frac{1}{2g^2} Tr\{\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\} \tag{146}$$

where we have summed over possible quark flavours, f. Very often, however, one prefers an alternative notation where the matrices $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$are expanded on the Lie algebra basis

$$\begin{aligned}\mathcal{A}_\mu(x) &= -igA^a_\mu(x)\mathcal{T}^a \\ \mathcal{F}_{\mu\nu}(x) &= -igF^a_{\mu\nu}(x)\mathcal{T}^a\end{aligned} \tag{147}$$

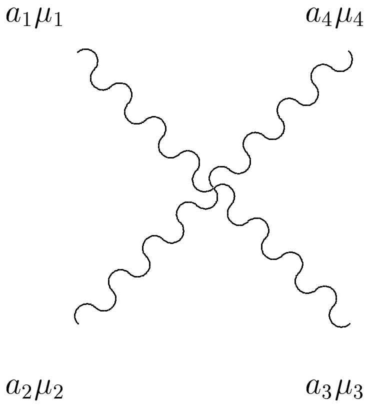
(implied sum over $a = 1, ..., N_c^2 - 1$ in $SU(N_C)$, i.e. over $a = 1, ..., 8$ in $SU(3)$). Using the expressions above it is then an easy matter to establish the following

$$\begin{aligned}F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \\ \mathcal{L}_{QCD} &= \sum_f \overline{q}^f (i \not{\partial} - m_f) q^f + g \sum_f \overline{q}^f \mathcal{T}^a \Omega_\mu q^f A^{\mu,a}\end{aligned}$$



$$
\begin{aligned}
= g f^{abc}[\eta^{\mu\rho}(k_c - k_a)^\nu + \eta^{\mu\nu}(k_a - k_b)^\rho \\
+ \eta^{\nu\rho}(k_b - k_c)^\mu]
\end{aligned}
$$

Fig. 4: 3-gluon vertex.



$$
\begin{aligned}
= -ig^2 \left[f_{a_1 a_2 b} f_{b a_3 a_4} \left(\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \right) \right. \\
+ f_{a_1 a_3 b} f_{b a_2 a_4} \left(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \right) \\
\left. + f_{a_1 a_4 b} f_{b a_2 a_3} \left(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_3} \right) \right]
\end{aligned}
$$

Fig. 5: 4-gluon vertex.

$$
\begin{aligned}
& - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} \\
\mathcal{L}_{YM} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,a} = \mathcal{L}_{2g} + \mathcal{L}_{3g} + \mathcal{L}_{4g} \\
\mathcal{L}_{2g} &= -\frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a}) \\
\mathcal{L}_{3g} &= -\frac{1}{2} g f^{abc} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) A^{\mu,b} A^{\nu,c} \\
\mathcal{L}_{4g} &= -\frac{1}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu,d} A^{\nu,e}
\end{aligned} \tag{148}
$$

The term $\mathcal{L}_{2g}$ is similar to the QED kinetic piece and will give rise to a gluon propagator equal to the photon propagator with a colour Kronecker δ^{ab} added. The terms $\mathcal{L}_{3g}$ and $\mathcal{L}_{4g}$ represent the principal new feature of non-abelian gauge theory and describe triple- and quadruple-gluon interactions. The corresponding Feynman rules are worked out from the rule given in subsection 2.3 with the results given in figs. 4 and 5. Just like in QED, in order to define the propagator we have to fix a gauge, make a gauge choice. For non-abelian gauge theory there is an important subtlety associated with that, which we only very briefly indicate. To impose a gauge condition like

$$\partial_\mu A^{\mu,a} = 0$$

say, we want somehow to insert a delta function in the path integral imposing that condition. This essentially implies changing integration variables from the gauge fields themselves A^a_μ, to something involving $\partial_\mu A^{\mu,a}$. Such a change of variable inevitably involves a Jacobian, here called a Fadeev-Popov determinant. In QED that determinant turns out to be independent of the field configurations, and thus may be absorbed in a normalization constant. But for non-abelian gauge theories this is not so, the determinant cannot be neglected. We might write it as an exponential of it's logarithm in order to attempt

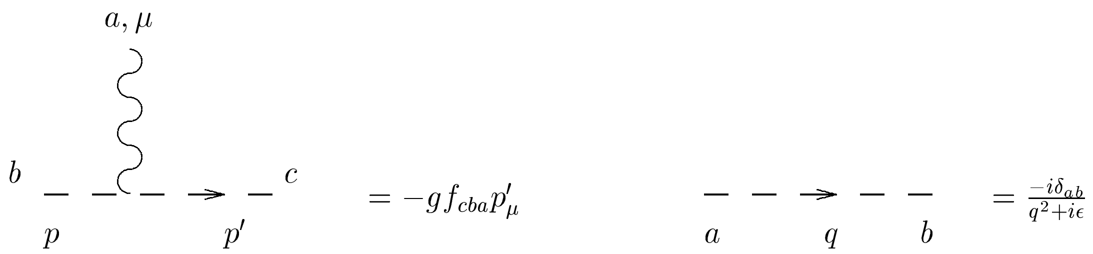


Fig. 6: Ghost-antighost-gluon vertex and ghost propagator in a covariant gauge.

treating it as a new term in the action, but that new term would be terribly non-local and it would not be possible to evaluate the path integral using Feynman rules. Fadeev and Popov noted, however, that any determinant may be written as a formal Gaussian path integral over Grassmann fields, with the operator of which we need the determinant standing between the fields. This is in contrast to the Gaussian integral over ordinary numbers which involves the inverse of a determinant. Using this trick, the resulting path integral takes the form of a more or less ordinary path integral with local action functions so that it may be evaluated by Feynman rules. The price is that we have been obliged to introduce the Fadeev-Popov ghosts, thus called because the Grassmann fields are Lorentz scalars and thus have the wrong spin-statistics relation. Since they merely represent a mathematical trick and not real particles, this is not a worry. It remains a fact that the formalism involves a Hilbert space which is somehow too large. The physical subspace has to be identified. The most elegant treatment is in terms of the so-called BRST operator, but a discussion of that goes beyond the scope of these notes. We finally remark that there does exist non-covariant gauges in which the Fadeev-Popov determinant is trivial, and ghosts may be forgotten about.

We finish by listing some good and some bad points about the classical QCD lagrangian we have established:

- Couplings between particles and gluons are given entirely by the representation matrix and one single coupling constant. In particular the gluon self-coupling is in terms of the structure constants f^{abc} which may be shown to be the relevant representation matrices of $\mathcal{T}^a$ in the so-called adjoint representation in which the gluons take their colour. Indeed, unlike quarks which are labelled by a colour index $i = 1, 2, ..., N_C$ with $N_C = 3$ for QCD, gluons are labelled by $a = 1, 2, ..., N_C^2 - 1$ with $N_C^2 - 1 = 8$ for QCD. This corresponds to the adjoint representation the dimension of which is the dimension of the Lie algebra. The matrix representation of the generator $\mathcal{T}^a$ is described by a matrix, the (b, c) element of which is $-if^{abc}$. Concerning the quarks it follows that they all couple with the same strength since they all lie in the same (triplet) representation of colour. This remarkable flavour independence tends to explain why flavour isospin invariance, for example is such a good symmetry. The flavour symmetry is broken by the non-equal quark masses.

- In the limit where quark masses may be neglected there is an enhanced chiral symmetry of the lagrangian. Namely the quark-gluon part may be written

$$\sum_f \bar{q}^f i \not{D} q^f = \sum_f \bar{q}^f_R i \not{D} q^f_R + \sum_f \bar{q}^f_L i \not{D} q^f_L$$
$$q = q_R + q_L = \frac{1}{2}(1 + \Omega_5) q + \frac{1}{2}(1 - \Omega_5) q \qquad (149)$$

  This implies the $U(N_f)_R \times U(N_f)_L$ invariance

$$q^f_R \to U(R)^{ff'} q^{f'}_R$$
$$q^f_L \to U(L)^{ff'} q^{f'}_L \qquad (150)$$

One part of this symmetry ($U(R) = U(L)$) appears to be realized in terms of approximately degenerate multiplets, whereas the other part ($U(R) = U(L)^{-1}$) appears to be spontaneously broken, the pion appearing as the approximate Goldstone boson for $N_f = 2$. In the quantum version of the theory it tunrs out that a $U(1)$ subgroup is anomalous, and the invariance group is only $SU(N_f)_R \times SU(N_f)_L \times U(1)$, the last $U(1)$ representing baryon number conservation.

- An apparently very bad feature of the theory is that (in the limit where quark masses are neglected) it depends on no dimensionful parameters (the coupling constant g is dimensionless as we have emphasized), whereas the strong interactions clearly know about a definite scale $\sim 100MeV - 200MeV$. Fortunately the quantum version of QCD remedies this in a nontrivial way. Due to renormalization, the coupling constant turns out to depend on the scale at which a given experiment is performed, introducing the concept of the "running coupling constant" which to one loop order is
$$\frac{g^2(Q^2)}{4\pi} = \frac{12\pi}{(33-2N_f)\log Q^2/\Lambda_{QCD}^2} \tag{151}$$
with Q^2 the scale in question and with Λ_{QCD} a new scale introduced by the quantization and renormalization. The conversion of a classical dimensionless parameter to a quantum dimensionful parameter is often denoted "dimensional transmutation". It will be studied in lectures 3 and 5. Further, eq.(151) implies that QCD is asymptotically free: for $Q^2 \to \infty$, $g^2(Q^2)/4\pi \to 0$.

3 LECTURE 3. 1-LOOP RENORMALIZATION OF $\lambda\phi^4$ THEORY

In this lecture we give an illustration of how loop calculations may be performed in the simplest case, and how the phenomenon of dimensional transmutation and the running coupling constant arises.

First a technical remark. The (scalar) propagators

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{p_0^2 - \vec{p}^2 - m^2 + i\epsilon}$$

have a cumbersome singularity when $p_0 = \pm\sqrt{\vec{p}^2 + m^2}$. The mathematical nuisance becomes smaller if one invokes the trick of carrying out a Wick rotation to imaginary energies. The point is that all amplitudes will turn out to be analytic functions of Lorentz invariants. Hence we may evaluate them for whatever unphysical values we please, and still obtain the physically relevant answers by analytic continuation. Thus we go to "euclidean" time and energy:

$$\begin{aligned} t &\to -it_E \\ E &\to +iE_E \\ iS(\phi) &\to -S_E = -\int d^4x_E \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right) \end{aligned} \tag{152}$$

There is no longer any need to distinguish between upper and lower "Lorentz" indices. And we change now the overall sign of the euclidean metric so that

$$p^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2$$

The euclidean Feynman rules are easily worked out to be given by fig.7

Our aim will be to look for quantum corrections to m and λ.

3.1 Propagator and coupling constant corrections, a first look

3.1.1 Propagator corrections

Every time we have a propagator in a Feynman diagram, we may consider the infinite series of Feynman diagrams which are identical to the first one, except that the propagator is "decorated" by extra loop

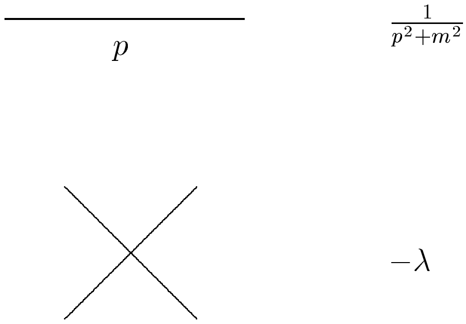


Fig. 7: Euclidean Feynman rules of ϕ^4 theory.

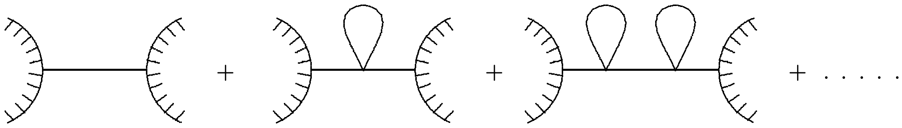


Fig. 8: Example of repeated single loop decorations of propagator in a Feynman diagram, represented by the “shaded parts”.

corrections. An example is given in fig. 8. More generally we have the situation shown in fig. 9, where the blobs, denoted $\Pi(p^2)$ (p is the euclidean 4-momentum flowing through the blob) are meant to represent the sum of all possible *1-particle irreducible* decorations. The term means that the decoration in question may *not* be separated in two disjoint pieces by cutting 1 single line. When we consider the sum of all such propagator decorations we reproduce the original Feynman diagram with the exception that the original propagator

$$\frac{1}{p^2+m^2}$$

is replaced according to

$$\begin{aligned} \frac{1}{p^2+m^2} &\to \frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2} + \frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2}\Pi(p^2)\frac{1}{p^2+m^2} + ... \\ &= \frac{1}{p^2+m^2-\Pi(p^2)} \end{aligned} \tag{153}$$

(remember $1+x+x^2+... = \frac{1}{1-x}$) Now consider the lowest order non trivial contribution to $\Pi(p^2)$ given by fig. 10:

$$\Pi(p^2) = -\lambda\frac{1}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2} \tag{154}$$

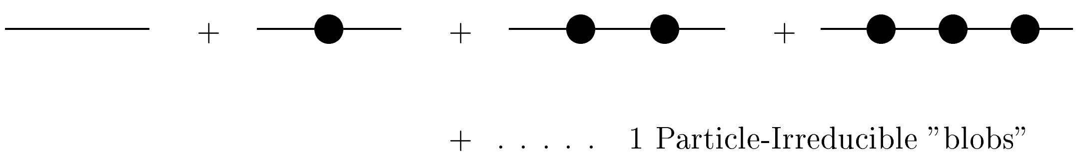


Fig. 9: Sum over 1-particle irreducible blobs in propagator correction.

The factor of $\frac{1}{2}$ comes from a weight factor which in this case is not just 1. In fact, we have to expand

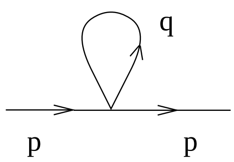


Fig. 10: 1-loop contribution to the 1-particle-irreducible propagator correction in ϕ^4 theory.

Dyson's formula to 1st order, so we start with $\frac{\lambda}{4!}$. Then we count the number of topologically equivalent contractions. The first of the 2 legs may be connected to any of the 4 legs of the vertex, giving a factor of 4. Then the second leg may be connected to any of the remaining 3, but after that the diagram may be completed in just one way. So The sought number is $4 \cdot 3 = \frac{1}{2} \cdot 4!$, whence the weight factor of $\frac{1}{2}$. The problem is that the integral in eq.(154) is badly - in fact quadratically - divergent. At the upper end of the integration we get approximately after introducing a cut-off:

$$\int^{\Lambda} q^3 dq \frac{1}{q^2} \sim \Lambda^2 \tag{155}$$

(which of course does have the correct dimension, namely the same as $p^2 + m^2$). So we have a problem to which we shall have to soon come back.

3.1.2 The 4-point amplitude to 1 loop order

The amplitude for the scattering $1 + 2 \to 3 + 4$ is given to 1-loop order by the diagrams in fig. 11. The

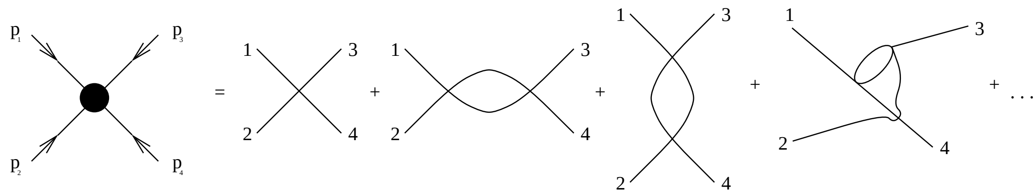


Fig. 11: The Feynman diagrams for the scattering amplitude of $1 + 2 \to 3 + 4$ to 1-loop order in ϕ^4 theory.

3 1-loop diagrams are entirely similar and it is enough to work out the first one. Again we get a weight factor of $\frac{1}{2}$by counting the number of possible ways of completing the diagram in fig. 12. The first

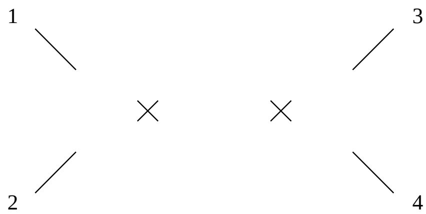


Fig. 12: The uncompleted Feynman diagram corresponding to the first 1-loop diagram in fig. 11. It may be completed in $8 \cdot 3 \cdot 4 \cdot 3 \cdot 2 = (4!)^2$ ways. Since it is second order in the coupling, that should be compared with the factor $\frac{1}{2!}\left(\frac{\lambda}{4!}\right)^2$, thus producing a weight factor of $\frac{1}{2}$.

1-loop diagram in fig. 11 is repeated in fig. 13 with momenta in the propagators indicated. Then we find for the corresponding part of the amplitude, to be denoted $A_s(p_1, p_2, p_3, p_4)$

$$A_s = \lambda^2 \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + m^2)((p+q)^2 + m^2)} \tag{156}$$

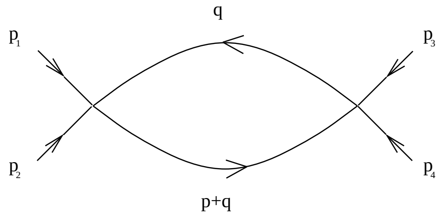


Fig. 13: The s-channel part of the 1-loop amplitudes for 4-particle scattering in ϕ^4 thory. The variable loop momentum is q. The other internal propagator has loop momentum $p+q$ according to 4-momentum conservation at vertices. Here $p \equiv p_1 + p_2 = -(p_3 + p_4)$.

where $p = p_1 + p_2 = -(p_3 + p_4)$. Again the loop integral diverges. Introducing the same sort of crude cut-off we find

$$\int^{\Lambda} q^3 dq \frac{1}{q^4} \sim \log \Lambda \tag{157}$$

So we must rethink completely what it is we are trying to do.

3.2 Renormalization

Let us very briefly indicate Wilson's intuition for what renormalization means. It is very useful to think in terms of the path integral

$$\int_{\text{All fields}} \mathcal{D}\phi e^{-S(\phi)} \tag{158}$$

(we think in terms of the Wick rotated or euclidean formulation). Obviously the concept of integrating over "all fields" is a very dubious one. It will include quite "crazy" fields that vary in arbitrarily wild manners. Perhaps it is better to think of the problem in two (or more) steps. We may imagine a split between field variations defined in terms of Fourier or momentum modes (the details are hopefully not too important), so that we first imagine integrating over all the wild modes with momenta larger than some cut-off, Λ, and giving rise to some effective action depending on that cut-off

$$\int_{\text{crazy modes}} \mathcal{D}\phi e^{-S(\phi)} \equiv e^{-S^{\Lambda}_{eff}(\phi)} \tag{159}$$

In the second step we should complete the path integral by using the effective action S^{Λ}_{eff} and *only integrating over slow modes* corresponding to momenta smaller than Λ. In fact we need not even imagine understanding precisely how the effective action is obtained as a result of integrating over the fast modes. All we need imagine is that it exists. If it exists we may use it to give a much better definition of our path integral, and we may use it to derive Feynman rules just as before, but with the important difference that in the new Feynman rules we should only carry out loop integrations for loop momenta smaller than the cut-off.

The crucial point about renormalizable theories turns out to be the following: The effective action looks almost exactly like our original action, but with the crucial modification that all the parameters, like mass and coupling constants acquire a dependence on the cut-off. In fact the normalization of the field itself does too. But when that is all that happens, we talk about a *renormalizable theory.* The loop integrals carried out up to the cutoff will not strictly diverge, but they will behave in a singular way as the cut-off is removed. However, the parameters, coupling constants etc. will also behave in a singular way, and precisely so that the net result is non-singular when the cut-off is removed. We thereby have a well-defined, renormalizable theory.

Thus we are faced with the following issues: (i) We must introduce a convenient cutoff in our Feynman loop integrals. (ii) We must accept to work with a modified, cut-off dependent action, called "the renormalized action" (a bad word, but this is the historical term) of the form

$$\mathcal{L}_{\text{ren}} = \frac{1}{2}\partial_\mu\phi_0\partial_\mu\phi_0 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4 \tag{160}$$

(iii) we must discover in the process of the calculation how in fact ϕ_0, m_0, λ_0 depend on the cutoff. One common practical way of doing that is to write the renormalized lagrangian as the original one plus a perturbation, and then attempt to find the à priori unknown perturbative part order by order in perturbation theory in λ (or whatever coupling constants will appear):

$$\begin{aligned}\mathcal{L}_{\text{ren}} &= \frac{1}{2}\partial_\mu\phi_0\partial_\mu\phi_0 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4 \\ &= \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\tilde{\lambda}}{4!}\phi^4 \\ &+ C\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + B\frac{1}{2}m^2\phi^2 + A\frac{\tilde{\lambda}}{4!}\phi^4 \\ &= \mathcal{L}_{\text{bare}} + \mathcal{L}_{\text{counter term}}\end{aligned} \tag{161}$$

Here m and λ are cut-off independent finite parameters characterizing the theory. A, B, C are functions of the cut-off and of these parameters. They are unknown to begin with, but will be determined in the process of the calculation order by order in λ. The meaning of $\tilde{\lambda}$ will become clear shortly.

3.3 Dimensional regularization

Many different regularizations schemes have been employed. Perhaps the most intuitive one is the lattice regularization in which the meaning of the cut-off is very clear. It has some interesting properties, and gives rise to a non-perturbative definition of QCD and provides a regularization that respects the gauge invariance of the theory. But of course it breaks Lorentz invariance very badly and it is not convenient for performing perturbative calculations based on Feynman diagrams. For this purpose dimensional regularization seems the most convenient scheme. It is the only one known which respects both Lorentz invariance and non-abelian gauge invariance. Unfortunately it rather lacks intuitive appeal. The idea is that the divergent integrals we have met, would in fact converge in lower space-time dimensions. One therefore attempts to define them as analytic functions of the dimension, well defined at first only for small unphysical dimensions, but subsequently *defined* by analytic continuation in dimension d. One then finds (of course) that the integrals exhibit a singularity (a pole) when $d = 4$. The regularization then consists in taking $d = 4 - \epsilon$ with ϵ somehow small. It is perhaps understandable that effectively this procedure is not too different from a momentum cut-off: Reducing the dimensionality of the momentum integration means somehow removing some volume at high momentum, which is what we do in a very crude way in a momentum cut-off. A crucial aspect of renormalizable theories is, that when defined by the limiting procedure mentioned, the limit is independent of the fine details of the choice of cut-off. This is rather like defining the derivative of a function. This universality has been checked in a number of cases.

3.3.1 The one loop master formula

We shall need a few simple facts about Euler's Gamma function

$$\Gamma(z) = \int_0^\infty d\alpha\alpha^{z-1}e^{-\alpha} = (z-1)! \tag{162}$$

This integral converges for any complex z such that $\Re z > 0$, and it generalizes the elementary factorial defined on integers. It is possible to extend the Gamma function by analytic continuation to the entire complex z plane, with the exception of $z = 0, -1, -2, -3, \ldots$ at which points the Gamma function has simple poles, in fact

$$\begin{aligned}
\Gamma(-n+\epsilon) &= \frac{(-)^n}{n!}[\frac{1}{\epsilon} + \psi(n+1) + \mathcal{O}(\epsilon)] \\
\psi(z) &\equiv \frac{d}{dz}\log\Gamma(z) \\
\psi(n) &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \Omega \\
\Omega &= 0.57721\ldots \quad \text{Euler's constant}
\end{aligned} \tag{163}$$

Now consider the integral in the propagator correction in d dimensions

$$\begin{aligned}
\int \frac{d^d q}{(2\pi)^d}\frac{1}{q^2+m^2} &= \int \frac{d^d q}{(2\pi)^d}\int_0^\infty d\alpha e^{-\alpha(q^2+m^2)} \\
&= \int_0^\infty d\alpha e^{-\alpha m^2}\frac{1}{(2\pi)^\alpha}\left(\frac{\pi}{\alpha}\right)^{d/2} = \frac{(m^2)^{d/2-1}}{(4\pi)^{d/2}}\Gamma(1-\frac{d}{2})
\end{aligned} \tag{164}$$

where we have employed gaussian integration to do the integral over q. We see that the result is an analytic function of d with poles at $d = 2, 4, 6, \ldots$. So as anticipated, we regain the original infinity of the integral in 4 dimensions. We obtain our master formula by differentiating $n-1$ times with respect to m^2

$$\int \frac{d^d q}{(2\pi)^d}\frac{1}{(q^2+m^2)^n} = \frac{\Gamma(n-\frac{d}{2})}{(4\pi)^n\Gamma(n)}\left(\frac{m^2}{4\pi}\right)^{d/2-n} \tag{165}$$

3.3.2 Mass renormalization

It seems that dimensional regularization is parametrized by a dimensionless parameter, d or ϵ (with $d = 4-\epsilon$) only. Actually we have to introduce a dimensionful one as well just as with a momentum cut-off. This may be seen by the following dimensional considerations. In d dimensions we may work out

$$\begin{aligned}
[\int d^d x\mathcal{L}] = 1 &\Rightarrow [\mathcal{L}] = E^d \\
[m^2\phi^2] = E^d &\Rightarrow [\phi] = E^{d/2-1} \\
[\tilde{\lambda}\phi^4] = E^d &\Rightarrow [\tilde{\lambda}] = E^\epsilon
\end{aligned} \tag{166}$$

Hence we want to write

$$\tilde{\lambda} = \mu^\epsilon\lambda \tag{167}$$

where μ is an arbitrary energy scale we must learn how to get rid of eventually.

Now

$$\begin{aligned}
\Pi(p^2) &= -\frac{1}{2}\mu^\epsilon\lambda\int\frac{d^d q}{(2\pi)^d}\frac{1}{q^2+m^2} = -\frac{\lambda}{2}\frac{\Gamma(\frac{\epsilon}{2}-1)}{(4\pi)^2}\left(\frac{4\pi\mu}{m^2}\right)^{\frac{\epsilon}{2}} m^2 \\
\Gamma(-1+\frac{\epsilon}{2}) &= -[\frac{2}{\epsilon} + \psi(2) + \mathcal{O}(\epsilon)] \Rightarrow \\
\Pi(p^2) &= \frac{\lambda}{(4\pi)^2}m^2\left(\frac{1}{\epsilon} + \text{finite}\right)
\end{aligned} \tag{168}$$

This is the result we obtain by dimensional regularization using the term $\mathcal{L}_{\text{bare}}$ in $\mathcal{L}_{\text{ren}}$. But we must remember also the contribution from the so-far unknown $\mathcal{L}_{\text{counter term}}$ in eq.(161). This term we see will contribute new Feynman rules when treated as a perturbation. The new rules will give new 2-leg vertices and a new 4-leg vertex. In particular the B term $B\frac{1}{2}m^2\phi^2$ will give a two leg vertex with the value $-Bm^2$. We refer to the rule in sect. 2.3 according to which we should work out the Fourier transform of the double derivative (for 2 legs) of $-S$ (in the euclidean case) with respect to ϕ. B is à priori unknown, but in order for the complete calculation to provide a finite result for $\Pi(p^2)$ the simplest possibility is to take

$$B = \frac{\lambda}{(4\pi)^2}\frac{1}{\epsilon} \tag{169}$$

Additional finite modifications are possible. This present prescription of making the counter terms simply remove the ϵ pole and nothing else, is called *minimal subtraction* or the MS-scheme. Different schemes will give rise to different physical meanings to the parameter λ but will otherwise describe exactly the same theory. We notice that our calculation of $\Pi(p^2)$ did *not* give rise to any p^2 term to 1-loop order. Hence we have no need for the C term in eq.(161), so

$$C = 0 \tag{170}$$

to first order in λ. So we have seen that the propagator correction $\Pi(p^2)$ may be treated at least to lowest order by renormalization theory: we need counter terms only of the kinds already present in the lagrangian.

3.3.3 Coupling constant renormalization

To understand the 1-loop correction to the amplitude for $1+2 \to 3+4$ figs. 11,13 requires more work. With our new notation in d dimension we find for the diagram fig. 13

$$A_s(\{p_i\}) = \frac{1}{2}\lambda^2\mu^{2\epsilon}\int\frac{d^dq}{(2\pi)^d}\frac{1}{[q^2+m^2][(q+p)^2+m^2]} \tag{171}$$

Unfortunately our master formula eq.(165) does not tell us how to carry out this integral. For this we somehow need to manipulate the integrand into the form

$$\frac{1}{(q^2+A^2)^2} \tag{172}$$

In fact it is possible to show that we may write

$$\int\frac{d^dq}{(2\pi)^d}\frac{1}{[q^2+m^2][(q+p)^2+m^2]} = \int\frac{d^dq}{(2\pi)^d}\int_0^1 d\alpha\frac{1}{[q^2+m^2-\alpha(1-\alpha)s]^2} \tag{173}$$

with

$$s = -(p_1+p_2)^2 \tag{174}$$

the usual Mandelstam variable (the minus sign is because we are in euclidean metric). This is of the form eq.(172) with $A^2 = m^2 - \alpha(1-\alpha)s$. So the master formula may be applied. To arrive at eq.(173) we have introduced a so-called Feynman- or Schwinger-parameter as follows:

Let $\Delta_1 \equiv q^2+m^2$ and $\Delta_2 \equiv (q+p)^2+m^2$. Eq.(173) follows from the following identity

$$\frac{1}{\Delta_1\Delta_2} = \int_0^\infty d\alpha\frac{1}{[(1-\alpha)\Delta_1+\alpha\Delta_2]^2} \tag{175}$$

Proof: write

$$\frac{1}{\Delta_1\Delta_2} = \int_0^\infty d\alpha_1 d\alpha_2 e^{-\alpha_1\Delta_1-\alpha_2\Delta_2} \tag{176}$$

Change variables: $\alpha_1 = t(1-\alpha), \alpha_2 = t\alpha, \alpha_1 + \alpha_2 = t$ with $t \in [0, \infty[$ and $\alpha \in [0,1]$. Integration over t then produces the desired result (using the Euler integral for the factorials, eq.(162)). Eq.(175) then gives the result eq.(173). In fact

$$\begin{aligned}(1-\alpha)\Delta_1 + \alpha\Delta_2 &= q^2 + 2\alpha q\cdot p + M^2 + \alpha p^2 \\ &= (q+\alpha p)^2 + \alpha(1-\alpha)p^2 + m^2 \\ &= q'^2 + m^2 - \alpha(1-\alpha)s \end{aligned} \tag{177}$$

So we see we also have to shift the integration variable q as $q \to q + \alpha p$. Then eq.(173) obtains.

Collecting pieces, we then get

$$A_s(\{p_i\}) = \frac{1}{2}\lambda^2\mu^{2\epsilon}\int_0^1 d\alpha \frac{\Gamma(\frac{\epsilon}{2})}{(4\pi)^2}\left(\frac{M^2 - \alpha(1-\alpha)s}{4\pi}\right)^{-\epsilon/2} \tag{178}$$

There are 2 more 1-loop diagrams obtained form this one by the substitutions $s \to u$ and $s \to t$ with $u = -(p_1+p_4)^2$ and $t = -(p_1+p_3)^2$. It is then a simple matter to sum these 3 contributions. We further expand for small ϵ:

$$\begin{aligned}\Gamma(\frac{\epsilon}{2}) &= \left(\frac{2}{\epsilon} + \psi(1) + \mathcal{O}(\epsilon)\right) \\ \psi(1) &= 1 - \varOmega \\ \left(\frac{4\pi\mu^2}{m^2}\right)^{\frac{\epsilon}{2}} &= 1 + \frac{\epsilon}{2}\log\frac{4\pi\mu^2}{m^2} + \mathcal{O}(\epsilon) \\ [1-\alpha(1-\alpha)\frac{s}{m^2}]^{-\frac{\epsilon}{2}} &= 1 - \frac{\epsilon}{2}\log[\alpha(\alpha-1)\frac{s}{m^2}] + \mathcal{O}(\epsilon)\end{aligned} \tag{179}$$

Hence the scattering amplitude for the process becomes (before the counter terms in eq.(161) are taken into account) summing all 3 loop terms

$$\begin{aligned}T_4 &= -\lambda\mu^\epsilon + \mu^\epsilon\frac{\lambda^2}{(4\pi)^2}\left(\frac{3}{\epsilon} + F(s,t,u,m,\mu,\epsilon)\right) \\ F(s,t,u,\mu,\epsilon) &= \frac{3}{2}(1 - \varOmega + \log 4\pi - \log\frac{m^2}{\mu^2}) \\ &- \frac{1}{2}\int_0^1 d\alpha\{\log(1-\alpha(1-\alpha)\frac{s}{m^2}) + (s\to t) + (s \to u)\}\end{aligned} \tag{180}$$

Exercise: Show that

$$\int_0^1 d\alpha \log(1-\alpha(1-\alpha)\frac{s}{m^2}) = -2 + \sqrt{1-\frac{4m^2}{s}}\log\left(\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1}\right) \tag{181}$$

We now have to include the contribution from the counter term $A\mu^\epsilon\frac{\lambda}{4!}\phi^4$. And we have to discover what A should be in order that the complete value for T_4 be finite when the cut-off ϵ is taken to 0. The counter term will give rise to a new 4-leg vertex, the value of which according our standard rule from sec. 2.3 will be $-A\mu^\epsilon\lambda$. We see that if we take (in the MS scheme)

$$A = \frac{3}{\epsilon}\frac{\lambda}{(4\pi)^2} \tag{182}$$

then we obtain a finite renormalized scattering amplitude. Let us notice that if we add to A a simple finite contribution so that F in eq.(180) is modified in such a way that the terms $-\varOmega + \log 4\pi$ are removed, then one talks about the $\overline{\text{MS}}$ scheme.

Let us summarize our findings

$$\begin{aligned}
\mathcal{L}_{\text{counter term}} &= \left(\frac{\lambda}{16\pi^2}\frac{1}{\epsilon}\right)\frac{1}{2}m^2\phi^2 + \mu^\epsilon\left(\frac{\lambda^2}{16\pi^2}\frac{3}{\epsilon}\right)\frac{\phi^4}{4!} \\
C &= 0; \quad B = \frac{\lambda}{(4\pi)^2}\frac{1}{\epsilon}; \quad A = \frac{\lambda}{(4\pi)^2}\frac{3}{\epsilon} \\
\phi_0 &= \phi \\
m_0^2 &= m^2\left(1 + \frac{\lambda}{(4\pi)^2}\frac{1}{\epsilon}\right) \\
\lambda_0 &= \mu^\epsilon\lambda\left(1 + \frac{\lambda}{(4\pi)^2}\frac{3}{\epsilon}\right)
\end{aligned} \tag{183}$$

3.4 The renormalization group equations

We have obtained a renormalized finite scattering amplitude, but we are not yet quite happy because the result depends on a strange "arbitrary" scale μ. We want to get a more convenient parametrization and interpretation. The finite scattering amplitude may be *either* considered to be a function of the renormalized parameters $T_4(\{p_i\}; m, \lambda, \mu, \epsilon)$, or if we prefer of the bare parameters $T_4(\{p_i\}, m_0, \lambda_0, \epsilon)$. The relation between the two sets is given by eq.(183). Also the two forms of T_4 are entirely identical:

$$\begin{aligned}
T_4(\{p_i\}, m_0, \lambda_0, \epsilon) &= T_4(\{p_i\}; m, \lambda, \mu, \epsilon) \\
&= \lambda + \mu^\epsilon\frac{\lambda^2}{(4\pi)^2}\left(\frac{3}{2}(\log 4\pi - \Omega) - \frac{3}{2}\log\frac{m^2}{\mu^2} + \frac{9}{2}\right. \\
&\left. - \left\{\frac{1}{2}\sqrt{1 - \frac{4m^2}{s}}\log\left(\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1}\right) + (s \to t) + (s \to u)\right\}\right)
\end{aligned} \tag{184}$$

Thus, depending of our taste we may consider one set of parameters or the other to be the dependent one, the functional relationship between the two in any case being given by eq.(183):

$$\begin{aligned}
T_4 &= T_4(\{p_i\}; m, \lambda, \mu, \epsilon) \\
T_4 &= T_4(\{p_i\}; m_0, \lambda_0, \epsilon) \\
\lambda_0 &= \lambda_0(\lambda, \mu, \epsilon) \\
m_0 &= m_0(\lambda, \mu, \epsilon) \\
\lambda &= \lambda(\lambda_0, \mu, \epsilon) \\
m &= m(\lambda_0, m_0, \mu, \epsilon)
\end{aligned} \tag{185}$$

A technical but important remark is that in the MS or the $\overline{\text{MS}}$ scheme λ is independent of m_0.

We now derive the renormalization group equations by using the fact that the scattering amplitude is independent of μ for fixed λ_0, i.e. when we use $(m_0, \lambda_0, \epsilon)$ as the independent set of variables. We act with

$$\mu\frac{d}{d\mu} = \frac{d}{d\log\mu}$$

and obtain

$$\begin{aligned}
0 &= \frac{d}{d\log\mu}T_4(\{p_i\}; m_0, \lambda_0, \epsilon) \\
&= \left(\frac{\partial}{\partial\log\mu} + \frac{\partial\lambda}{\partial\log\mu}\frac{\partial}{\partial\lambda} + \frac{\partial\log m}{\partial\log\mu}\frac{\partial}{\partial\log m}\right)T_4(\{p_i\}; m, \lambda, \mu, \epsilon)
\end{aligned} \tag{186}$$

Denote

$$\beta(\lambda,\epsilon) \equiv \frac{d\lambda}{d\log\mu}|_{\text{fixed}\lambda_0,\epsilon}$$
$$\Omega_m(\lambda,\mu,\epsilon) \equiv \frac{d\log m}{d\log\mu}|_{\text{fixed}\lambda_0,\epsilon} \tag{187}$$

where we have introduced the notion of the *beta function* and the mass dimension.

Let us find the beta function to 1-loop order for the ϕ^4 theory from our calculation. We act with $\frac{d}{\log\mu}$ on

$$\lambda_0 = \lambda\mu^\epsilon\left(1+\frac{\lambda}{(4\pi)^2}\frac{3}{\epsilon}\right) \tag{188}$$

giving

$$\begin{aligned} 0 &= \epsilon\lambda\left(1+\frac{\lambda}{(4\pi)^2}\frac{3}{\epsilon}\right)+\beta(\lambda,\epsilon)\left(1+\frac{2\lambda}{(4\pi)^2}\frac{3}{\epsilon}\right) \Rightarrow \\ \beta(\lambda,\epsilon) &= -\epsilon\lambda\frac{1+\frac{\lambda}{(4\pi)^2}\frac{3}{\epsilon}}{1+\frac{6\lambda}{(4\pi)^2}\frac{1}{\epsilon}} \\ &\simeq -\epsilon\lambda+3\frac{\lambda^2}{(4\pi)^2}+\mathcal{O}(\lambda^3) \end{aligned} \tag{189}$$

So finally

$$\beta(\lambda) = +3\frac{\lambda^2}{(4\pi)^2} \tag{190}$$

Notice that in this calculation we have made expansions order by order in λ for fixed ϵ. In other words, we have considered the limit $\lambda \to 0$ *before* the limit $\epsilon \to 0$.

3.4.1 Scaling and the running coupling constant

It will be convenient to find an improved version of the renormalization group equation eq.(186) in order to get rid of the unknown μ. We shall use a somewhat simplified treatment here avoiding to introduce greens functions and continue to talk about scattering amplitudes. The (small) problem will be that for the scattering amplitude $p_i^2 = -m_i^2$ (in the euclidean case) whereas we shall really want to consider external momenta not on this mass shell. Forgetting this subtlety we first notice that T_4 is dimensionless (the lowest order contribution is simply $-\lambda$). Hence it will be invariant under a simultaneous scaling of all dimensionful parameters:

$$p_i \to tp_i,\;\; m \to tm_i,\;\; \mu \to t\mu$$

Also notice that $\log(tm) = \log t + \log m$. Hence we find the trivial scaling property

$$\begin{aligned} \frac{d}{d\log t}T_4(\{tp_i\};tm,\lambda,t\mu) &= 0 \Rightarrow \\ \left(\frac{\partial}{\partial\log t}+\frac{\partial}{\partial\log m}+\frac{\partial}{\partial\log\mu}\right)T_4(\{tp_i\};m,\lambda,\mu) &= 0 \end{aligned} \tag{191}$$

Eliminating $\partial/\partial\log\mu$ between that and eq.(186), we get our final renormalization group equation

$$\left(-\frac{\partial}{\partial\log t}+\beta(\lambda)\frac{\partial}{\partial\lambda}+(\Omega_m-1)\frac{\partial}{\partial\log m}\right)T_4(\{tp_i\};m,\lambda,\mu) = 0 \tag{192}$$

This equation is solved by

$$T_4(\{tp_i\};m,\lambda,\mu) = T_4(\{p_i\};m(t),\lambda(t)) \tag{193}$$

where the running coupling constant and the running mass are solutions of the equations

$$\frac{d\lambda(t)}{d\log t} = \beta(\lambda(t)); \quad \frac{d\log m(t)}{d\log t} = \Omega_m(\lambda(t), m(t)) - 1 \tag{194}$$

In fact this follows trivially from

$$\frac{d}{d\log t} T_4(\{p_i\}; m(t), \lambda(t)) = \left(\frac{d\lambda(t)}{d\log t}\frac{\partial}{\partial\lambda} + \frac{d\log m(t)}{d\log t}\frac{\partial}{\partial\log m}\right) T_4(\{p_i\}; m(t), \lambda(t)) \tag{195}$$

which is just eq.(192).

Let us finally find the famous running coupling constant for the ϕ^4 theory by solving eq.(194):

$$\begin{aligned}
\frac{d\lambda(t)}{d\log t} &= \beta(\lambda(t)) = 3\frac{\lambda^2(t)}{(4\pi)^2} \Rightarrow \\
\frac{d}{d\log t}\frac{1}{\lambda(t)} &= -\frac{d\lambda(t)}{d\log t}\frac{1}{\lambda^2(t)} = -\frac{3}{(4\pi)^2} \Rightarrow \\
\frac{1}{\lambda(t)} &= -\frac{3}{(4\pi)^2}\log t/t_0 \Rightarrow \\
\lambda(t) &= -\frac{(4\pi)^2}{3\log t/t_0}
\end{aligned} \tag{196}$$

In particular, expressing the running coupling constant as a function of a typical momentum transfer (squared) type of scale, we would write

$$\lambda(q^2) = -\frac{32\pi^2}{3\log(q^2/\Lambda^2_{\phi^4})} \tag{197}$$

We see that all of a sudden the theory depends on some *characteristic scale* Λ_{ϕ^4} analogous to the famous Λ_{QCD} whereas the classical theory only depended on a *dimensionless* coupling constant. This is the phenomenon of dimensional transmutation.

In the present case we found a *positive* beta function. That resulted in the minus sign in front of eq.(197). That sign again tells us that the calculation only makes sense provided we are the infrared regieme where $q^2 \ll \Lambda^2_{\phi^4}$, so that the logarithm gets large and negative and the running coupling small and positive so that perturbation theory makes sense. This of course is opposite to QCD, and we shall come back to QCD in lecture 5.

Equation (192) tells us that we may perform reliable perturbative calculations when we "scale" to small (in this case) values of $|tp_i|$. The result is obtained by using the running coupling constant (and the running mass).

4 LECTURE 4. THE INFRARED PROBLEM IN QED

The famous infrared problem in QED is one that essentially only arises when we ask meaningless questions. Classically any scattering process involving charged particles will be associated with electromagnetic radiation having a spectrum with an infrared tail involving infinitely many photons. Thus it is not really surprising that quantum amplitudes involving a definite number of produced particles will have something strange about them. In particular there will be meaningless "infinities" of probabilities for such processes, associated with the fact that the quantum calculation is somehow trying desperately to reproduce the classical phenomenon involving an infinity of produced photons. The solution to the problem may be presented at any given order in the fine structure constant α. It involves the production of infrared photons not registered by the detector, and it involves certain radiative corrections to do with loop amplitudes. Both of these effects must be taken into account. When they are, a consistent picture emerges at any given order in α. An even more satisfactory result is obtained in a so-called leading log approximation, where certain diagrams involving arbitraryly high orders in α may be taken into account in some approximation. For that we shall merely state the result, the so-called Sudakov form factor. The treatment in this lecture is based rather closely on [5] Ch. 6, on [6] sects. 1-3-2 and 4-1-2 and on [3]. Very similar treatments may be found in several other books.

We shall concentrate on the example illustrated in fig. 14 describing the scattering of a single electron off a "heavy particle" for which the amplitude may be obtained to lowest order as

$$iT_{fi} = \overline{u}(p')(-ie\Omega_\mu)u(p)\frac{-i}{q^2}(\overline{u}(-ie\Omega^\mu)u)_{\text{heavy}} \tag{198}$$

Most of the time we shall not be interested in the structure involving the heavy particle, and we shall

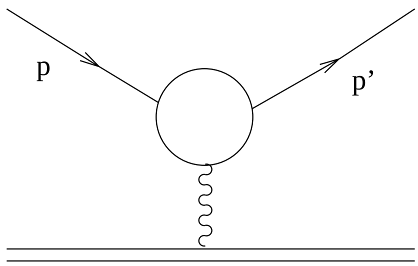


Fig. 14: Graph representing the sum of all Feynman diagrams for scattering of an electron with 4-momentum p off a heavy particle indicated by the double line, to become a new electron with 4-momentum p'. Only single photon exchange is taken into account, and only vertex corrections will be considered.

replace the part

$$\frac{-i}{q^2}(\overline{u}(-ie\Omega^\nu)u)_{\text{heavy}} \to A^\mu_{Cl}(q) \tag{199}$$

describing the scattering of the electron off a certain external "classical" field, $A^\mu_{Cl}(x)$ with Fourier transform $A^\mu_{Cl}(q)$, where

$$q^\mu \equiv (p' - p)^\mu$$

and with $A^\mu_{Cl}(q) \propto (2\pi)\delta(p'_0 - p_0)$ for a static external field.

Eq.(198) gives the result to lowest order in the electron charge. In general we would have a replacement

$$i(2\pi)\delta(p'_0 - p_0)T_{fi} \to \overline{u}(p')(-ie\Gamma_\mu(p,p'))u(p)A^\mu_{Cl}(q) \tag{200}$$

describing the sum of all diagrams contributing to the process, see fig. 15. In addition there will be

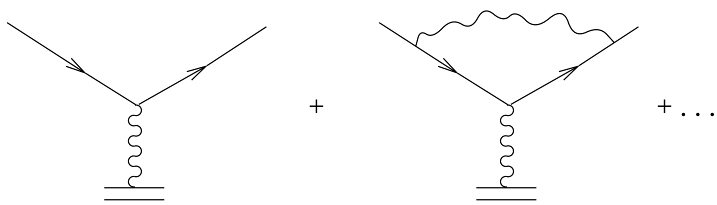


Fig. 15: The contributions to the two lowest orders for fig. 14.

mass renormalization diagrams and vacuum polarization diagrams which we shall not consider. They contribute to different physical effects. The order α contribution to the cross section comes from the interference between the two diagrams in fig. 15. The result will turn out to be of the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} f_{IR}(q^2) \log\left(\frac{-q^2}{\mu^2}\right) + ...\right] \tag{201}$$

Here we have been obliged to introduce a "photon mass" parameter μ by the substitution $\frac{1}{q^2} \to \frac{1}{q^2-\mu^2}$ in the photon propagator. Clearly this expression behaves in a singular unacceptable way when $\mu \to 0$.

This illustrates the problem of obtaining bad answers to bad questions. The question involves cross section for the scattering of an electron *without* any associated photon radiation. This is a physically unreasonable question. There will necessarily be some very soft photon emission that the detector might not be able to register and which would have to be included in a physically sensible counting rate. Thus we also want to consider the production of a single photon according to fig. 16. That will turn out to solve the problem.

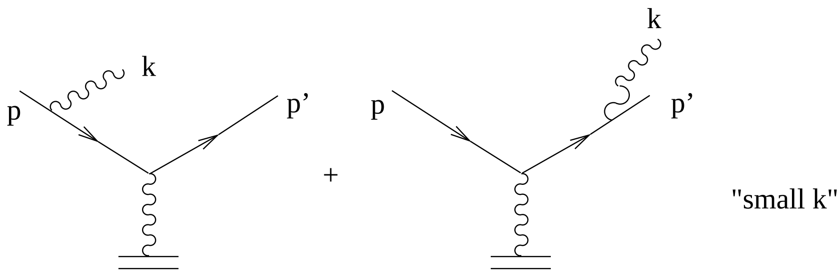


Fig. 16: Lowest order (soft) bremsstrahlung radiation diagrams.

4.1 The classical radiation problem

It is very instructive to work out the classical radiation associated with a deflected charged point particle. We consider the situation in fig. 17 where the particle experiences a sudden kick at $t = 0, \vec{x} = \vec{0}$ resulting in a change in 4-momentum from p^μ to p'^μ. Of course treating the momentum change as

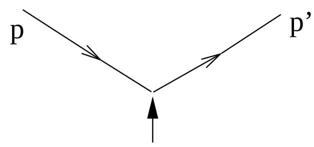


Fig. 17: Diagram for the "classical" scattering of an electron off an external field. The "sudden kick" approximation is implied. At the position of the arrow, $t = 0, \vec{x} = \vec{0}$ an instantaneous momentum transfer of $q^\mu = (p' - p)^\mu$ is given to the electron.

instantaneous is unrealistic and would require infinite force. We shall meet some mild disease arising from that approximation. It is immediately regulated by introducing a finite time interval Δt for the 4-momentum change. Quantum mechanically we expect $\Delta t \sim 1/|q|$.

The classical calculation is extremely straight forward. The 4-current density is given by the particle trajectory $x^\mu = y^\mu(\tau)$ (τ = proper time)

$$j^\mu(x) = e\int d\tau \frac{dy^\mu(\tau)}{d\tau}\delta^4(x^\mu - y^\mu(\tau)) \tag{202}$$

where the 4-velocity in our case is given by

$$\frac{dy^\mu(\tau)}{d\tau} = \begin{cases} p^\mu/m & \tau < 0 \\ p'^\mu/m & \tau > 0 \end{cases} \tag{203}$$

So

$$j^\mu(x) = e\int_0^\infty d\tau \frac{p'^\mu}{m}\,\delta^4(x-\frac{p'}{m}\tau) + e\int_{-\infty}^0 d\tau \frac{p^\mu}{m}\delta^4(x-\frac{p}{m}\tau) \tag{204}$$

with Fourier transform

$$j^\mu(k) = \int d^4x e^{ikx} j^\mu(x) = ie\left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon}\right) \tag{205}$$

the $\pm i\epsilon$'s being introduced to provide convergence. We can find $A^\mu(x)$ or $A^\mu(k)$ now. We impose the Lorentz gauge $\partial_\mu A^\mu = 0$ and find

$$\begin{aligned} A^\mu(k) &= -\frac{1}{k^2} j^\mu(k) \\ A^\mu(x) &= -ie\int \frac{d^4k}{(2\pi)^4} e^{-ikx}\frac{1}{k^2}\left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon}\right) \end{aligned} \tag{206}$$

The photon propagator this time will not be regulated according to the Feynman prescription, but according to one which will impose boundary conditions reasonable for the classical problem at hand. We shall want the so-called retarded propagator relevant when we want there to be no radiation prior to the kick of the charged particle. This is achieved by shifting both poles at $k_0 = \pm|\vec{k}|$ into the lower half plane. In that case we may close the k_0-integration contour for $t < 0$ in the upper half plane without any contribution from these poles. There are also the poles at

$$\begin{aligned} k^0 p'^0 &= \vec{k}\cdot\vec{p}' - i\epsilon \\ k^0 p^0 &= \vec{k}\cdot\vec{p} + i\epsilon \end{aligned} \tag{207}$$

For $t < 0$ we thus close in the upper half k^0 plane and get the contribution from the last of those poles (in the $\vec{p} = \vec{0}$ frame)

$$A^0(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}\frac{e}{|\vec{k}|^2} \tag{208}$$

which is just the Coulomb potential from the charged particle at rest for $t < 0$ (exercise: show that). We neglect that (and the similar term from the final particle) since we are only interested in the resulting radiation. For the radiation part we get

$$\begin{aligned} A^\mu_{rad}(\vec{k}) &= -2e\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right) \\ A^\mu_{rad}(x) &= \Re\int \frac{d^3k}{(2\pi)^3 2k^0} A^\mu_{rad}(\vec{k}) e^{-ikx} \\ \vec{E}(x) &= \Re\int \frac{d^3k}{(2\pi)^3 2k^0}\vec{E}(\vec{k}) e^{-ikx} \\ \vec{B}(x) &= \Re\int \frac{d^3k}{(2\pi)^3 2k^0}\vec{B}(\vec{k}) e^{-ikx} \\ \vec{E}(\vec{k}) &= -i\vec{k}A^0(\vec{k}) + ik^0\vec{A}(\vec{k}) \\ \vec{B}(\vec{k}) &= i\vec{k}\times\vec{A}(\vec{k}) = \hat{k}\times\vec{E}(\vec{k}) \\ \text{Energy} &= \frac{1}{2}\int d^3x(|\vec{E}|^2 + |\vec{B}|^2) = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2k^0}\vec{E}(\vec{k})\cdot\vec{E}^*(\vec{k}) \end{aligned} \tag{209}$$

Transversality $\vec{k}\cdot\vec{E}(\vec{k}) = 0$ implies that we may choose two polarization vectors for the radiation, $\vec{\epsilon}_1 \perp \hat{k}, \vec{\epsilon}_2 \perp \hat{k}, \vec{\epsilon}_1 \perp \vec{\epsilon}_2, |\vec{\epsilon}_i| = 1$ such that $\vec{E}(\vec{k}) = \vec{\epsilon}_1(\vec{k})(\vec{\epsilon}_1 \cdot \vec{E}) + \vec{\epsilon}_2(\vec{k})(\vec{\epsilon}_2 \cdot \vec{E})$. Hence

$$\text{Energy} = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} \frac{e^2}{2} \left| \vec{\epsilon}_\lambda(\vec{k}) \cdot \left(\frac{\vec{p}'}{k\cdot p'} - \frac{\vec{p}}{k\cdot p} \right) \right|^2 \tag{210}$$

Now

$$\sum_{\lambda=1}^{2} \epsilon^i_\lambda(\vec{k}) \epsilon^j_\lambda(\vec{k}) = \delta^{ij}_\perp \equiv \delta^{ij} - \hat{k}^i \hat{k}^j \tag{211}$$

In a frame where $k^\mu = (k^0, 0, 0, k^0)$ we may take

$$\begin{aligned}
\epsilon^\mu_1 &= (0,1,0,0) \\
\epsilon^\mu_2 &= (0,0,1,0) \\
\epsilon^\mu_3 &= (0,0,0,1) \\
\epsilon^\mu_0 &= (1,0,0,0)
\end{aligned} \tag{212}$$

Since

$$k^\mu \left(\frac{p'_\mu}{k\cdot p'} - \frac{p_\mu}{k\cdot p} \right) = 0 \tag{213}$$

the contribution from the unphysical photons with polarizations $\epsilon^\mu_0, \epsilon^\mu_3$ satisfies

$$(\epsilon^\mu_3 + \epsilon^\mu_0) \left(\frac{p'_\mu}{k\cdot p'} - \frac{p_\mu}{k\cdot p} \right) = 0 \tag{214}$$

so that we may use

$$\sum_{\lambda=0}^{3} \epsilon^\mu_\lambda(k) \epsilon^\nu_\lambda(k)^* = -\eta^{\mu\nu} \tag{215}$$

and (introducing $\hat{k}^\mu$ by $k^\mu = k^0 \hat{k}^\mu$)

$$\begin{aligned}
\text{Energy} &= \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{2} \left(\frac{2pp'}{(kp')(kp)} - \frac{m^2}{(kp')^2} - \frac{m^2}{(kp)^2} \right) \\
&= \frac{1}{(2\pi)^3} \int_0^\infty dk^0 d\Omega_{\hat{k}} \left(\frac{2pp'}{(\hat{k}p')(\hat{k}p)} - \frac{m^2}{(\hat{k}p')^2} - \frac{m^2}{(\hat{k}p)^2} \right) \\
&= \frac{e^2}{(2\pi)^2} \int_0^\infty dk^0 I(\vec{v}, \vec{v}') \\
I(\vec{v}, \vec{v}') &\equiv \int \frac{d\Omega_{\hat{k}}}{4\pi} \left(\frac{2(1 - \vec{v}\cdot\vec{v}')}{(1 - \hat{k}\cdot\vec{v})(1 - \hat{k}\cdot\vec{v}')} - \frac{m^2/E^2}{(1 - \hat{k}\cdot\vec{v}')^2} - \frac{m^2/E^2}{(1 - \hat{k}\cdot\vec{v})^2} \right)
\end{aligned} \tag{216}$$

where we use a frame such that

$$p^\mu = E(1, \vec{v}); p'^\mu = E(1, \vec{v}')$$

We see that the bremsstrahlung radiation predominately is emitted along the directions of the incoming or the outgoing charged particle. Namely, the first term in the integrand for $I(\vec{v}, \vec{v}')$ is nearly singular when $\hat{k} \approx \vec{v}$ or $\hat{k} \approx \vec{v}'$. In fact, denoting the angle between $\hat{k}$ and $\vec{v}$ (and similarly for $\vec{v}'$) by θ, the angular integration defining $I(\vec{v}, \vec{v}')$ is an integration over $d\cos\theta$ which is nearly log-behaved. The divergent log is cut off at $\theta = 0$ or $\cos\theta = 1$ at one end, and at something like $\hat{k}\cdot\vec{v} \approx \vec{v}\cdot\vec{v}'$ at the other end (the details are not important in leading log approximation). This gives the approximate leading log estimate of $I(\vec{v}, \vec{v}')$

$$I(\vec{v}, \vec{v}') \sim \log \frac{(1 - \vec{v} \cdot \vec{v}')^2}{(1 - |\vec{v}|^2)} = \log \frac{(E^2 - \vec{p} \cdot \vec{p}')^2}{E^2(E - |\vec{p}|)^2}$$
$$\approx 2 \log \frac{pp'}{2E(E - |\vec{p}|)} \approx 2 \log \frac{pp'}{E^2 - \vec{p}^2} \approx \log \left(\frac{-q^2}{m^2} \right) \tag{217}$$

The k^0 integral for the energy diverges at the upper end since $I(\vec{v}, \vec{v}')$ is completely independent of k^0. As we have anticipated this is to be expected from the instantaneous kick approximation. We repair it by integrating up to some $k_{max} \sim \sqrt{-q^2}$ only.

$$\text{Energy} = \frac{\alpha}{\pi} \int_0^{k_{max}} dk^0 I(\vec{v}, \vec{v}') \sim \frac{2\alpha}{\pi} k_{max} \log \left(\frac{-q^2}{m^2} \right) \tag{218}$$

Although this calculation is classical, we may use it to derive the photon multiplicity spectrum:

$$dn_\Omega = \frac{\alpha dk^0}{\pi \, k^0} \, I(\vec{v}, \vec{v}') \tag{219}$$

which diverges at $k^0 \to 0$, but as already emphasized that is no worry at all. There are just lots and lots of extremely soft photons, building an essentially classical field.

This finishes the classical calculation.

4.2 Single photon emission

Now let us compare the classical calculation with a quantum calculation of bremsstrahlung to lowest order in α where only a single emitted photon is involved. We shall find an unacceptable diverging probability which however, will be capable of canceling a similar trouble in the radiative correction to the zero photon emission problem, eq.(201) to be worked out in the next subsection.

The relevant Feynman diagrams are shown in fig. 18 with the result

$$iT_{fi} = \overline{u}(p') \Big\{ -ie \not{A}_{Cl}(p' - p + k) \frac{i(\not{p} - \not{k} + m)}{(p - k)^2 - m^2} (-ie \not{\epsilon}^*)$$
$$+ (-ie \not{\epsilon}^*) \frac{i(\not{p}' + \not{k} + m)}{(p' + k)^2 - m^2} (-ie \not{A}_{Cl}(p' + k - p)) \Big\} u(p) \tag{220}$$

We shall use the calculation only for soft photons so we put $A^\mu_{Cl}(p' - p + k) \simeq A^\mu_{Cl}(q)$ and we ignore

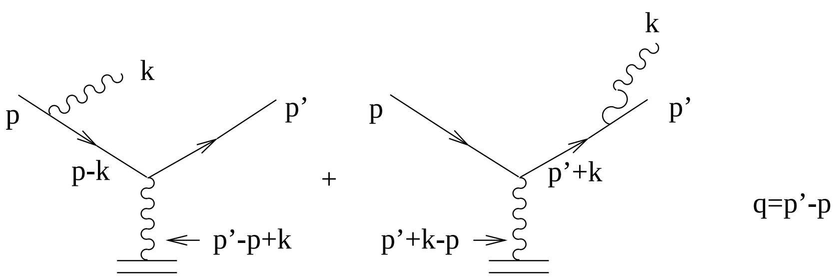


Fig. 18: Feynman diagrams with momentum labelling for lowest order bremsstrahlung.

$\not{k}$ in the numerators. Then we may work out

$$(\not{p} - \not{k} + m) \not{\epsilon}^* u(p) \sim (\not{p} + m) \not{\epsilon}^* u(p)$$
$$= \not{\epsilon}^* (-\not{p} + m) u(p) + 2p \cdot \epsilon^* u(p)$$
$$= 0 + 2p \cdot \epsilon^* u(p) \tag{221}$$

(using $\{\Omega_\mu, \Omega_\nu\} = 2\eta_{\mu\nu}$), and

$$
\begin{aligned}
iT_{fi} &= \{\overline{u}(p')(-ie\not{A}_{Cl}(q))u(p)\}\left\{ie\left(\frac{p\cdot\epsilon^*}{p\cdot k} - \frac{p'\cdot\epsilon^*}{p'\cdot k}\right)\right\} \\
d\sigma(p\to p'+\Omega) &= d\sigma_0(p\to p')\frac{d^3k}{(2\pi)^3 2k^0}e^2\sum_{\lambda=1}^{2}\left|\frac{p\cdot\epsilon_\lambda}{p\cdot k} - \frac{p'\cdot\epsilon_\lambda}{p'\cdot k}\right|^2
\end{aligned}
\tag{222}
$$

Here $d\sigma_0(p\to p')$ is the lowest order cross section for electron scattering without photon emission. The last factor describes the probability for photon emission. This result may be compared with the classical expression for the energy, eq.(210). In fact the two expressions agree when the above quantum probability is multiplied by k^0 to give the energy. However, dividing the classical energy by k^0 gave an acceptable divergent photon *multiplicity*, whereas in the present case it gives an unacceptable divergent single photon emission *probability*. When the full probability is to be calculated, we regulate the divergent integral over k^0 both at the upper and at the lower end:

$$
\int_0^\infty \frac{dk^0}{k^0} \to \int_\mu^{|q|} \frac{dk^0}{k^0} \sim \log\left(\frac{-q^2}{\mu^2}\right) \tag{223}
$$

The upper limit is a phase space effect here, not an approximation. At the lower end we have introduced a "photon mass" μ in order to get a finite integral. The probability is unacceptably divergent in the limit $\mu\to 0$. That difficulty will have to be solved. Doing the integral gives

$$
\begin{aligned}
d\sigma(p\to p'+\Omega) &= d\sigma_0(p\to p')\frac{\alpha}{2\pi}\log\left(\frac{-q^2}{\mu^2}\right)I(\vec{v},\vec{v}') \\
&\sim d\sigma_0(p\to p')\frac{\alpha}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\log\left(\frac{-q^2}{m^2}\right)
\end{aligned}
\tag{224}
$$

for $\log(-q^2)\to\infty$, showing the so-called Sudakov double logarithm.

4.3 The vertex correction

We now start on the somewhat harder work of understanding the order α vertex correction alluded to in eq.(201). We wrote in general to any order eq.(200)

$$
iT_{fi}(2\pi)\delta(p'_0 - p_0) = \overline{u}(p')(-ie\Gamma_\mu(p,p'))u(p)A^\mu_{Cl}(q) \tag{225}
$$

where we know that

$$
\begin{aligned}
\Gamma_\mu(p,p') &= \Omega_\mu + \delta\Gamma_\mu(p,p') \\
\delta\Gamma_\mu(p,p') &= \mathcal{O}(\alpha)
\end{aligned}
\tag{226}
$$

The form of this 4×4 matrix is constrained by Lorentz invariance and current conservation: $q^\mu\Gamma_\mu(p,p') = 0$. Namely, in order to transform as a 4-vector we must have

$$
\Gamma_\mu(p,p') = \Omega_\mu A(q^2) + (p_\mu + p'_\mu)B(q^2) + (p'_\mu - p_\mu)C(q^2) \tag{227}
$$

and current conservation is automatically satisfied for the first two terms but violated for the last, so $C(q^2)\equiv 0$. Also there is the Gordon identity

$$
i\sigma^{\mu\nu}q_\nu = 2m\Omega^\mu - (p'+p)^\mu \tag{228}
$$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\Omega^\mu, \Omega^\nu]$. This is proven by using $\overline{u}(p')\ \not{p}' = m\overline{u}(p')$ etc. and $\Omega^\mu\ \not{p}' = -\ \not{p}'\Omega^\mu + 2p^\mu$ from the commutation relations $\{\Omega^\mu, \Omega^\nu\} = 2\eta^{\mu\nu}$. It is then costumary to write the vertex $\Gamma_\mu(p,p')$ in terms of the two so-called electron form factors

$$\Gamma_\mu(p,p') = \Omega_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q^\nu}{2m}F_2(q^2) \tag{229}$$

To lowest order

$$F_1^{(0)} = 1, \quad F_2^{(0)} = 0 \tag{230}$$

Also, the normalization of the electron electric charge is defined by the condition

$$F_1(0) = 1 \tag{231}$$

$F_2(0)$ is related to the electron magnetic moment $F_2(0) = \frac{1}{2}(g-2)$ (see exercise, sect. 4.3.3).

4.3.1 The 1 loop evaluation of the vertex correction

With the notation provided by fig. 19 we obtain (in the Feynman gauge)

$$\begin{aligned}
\delta\Gamma^\mu(p,p') &= \int \frac{d^4k}{(2\pi)^4}\overline{u}(p')(-ie\Omega^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\Omega^\mu\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\Omega^\rho)u(p') \\
&\times \frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon} \\
&= -ie^2\overline{u}(p')\left\{\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)((k-p)^2+i\epsilon)}\right. \\
&\times \left.\Omega^\nu(\not{k}'+m)\Omega^\mu(\not{k}+m)\Omega_\nu\right\}u(p)
\end{aligned} \tag{232}$$

This integral is log divergent and we must carry out a (dimensional) regularization, even though in fact

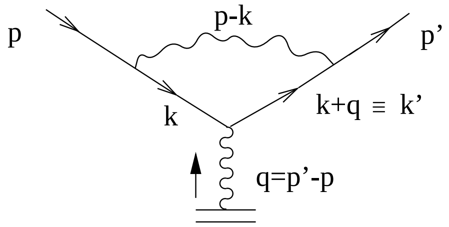


Fig. 19: Momentum labelling in the 1-loop vertex correction Feynman diagram.

the infra red problems we attempt to study will be associated with a (ultra violet) finite part of the integral. We cannot use directly our master formula for 1 loop integration eq.(165) until we have (i) introduced Feynman parameters and (ii) carried out a shift in the momentum integration.

In lecture 3 we met the Feynman parameter trick in the case of two propagators. Here we have three propagators. We therefore use the following:

$$\begin{aligned}
\frac{1}{D_1D_2D_3} &= \int_0^1 dxdydz\,\delta(x+y+z-1)\frac{2}{D^3} \\
D &= xD_1 + yD_2 + zD_3 \\
D_1 &= k^2 - m^2 + i\epsilon;\ \ D_2 = k'^2 - m^2 + i\epsilon;\ \ D_3 = (k-p)^2 + i\epsilon
\end{aligned} \tag{233}$$

Proof: We do the general case in fact and write

$$\frac{1}{D_1 \ldots D_n} = \int_0^\infty \prod_{i=1}^n d\xi_i e^{-(\xi_1 D_1 + \ldots + \xi_n D_n)} \tag{234}$$

Put $\xi_i = tx_i, \;\; t = \sum_i \; \xi_i, \;\; \sum_i x_i = 1$. Do the integral over t, using

$$\Gamma(n) = \int_0^\infty \; dt t^{n-1} e^{-t} = (n-1)!$$

to obtain

$$\frac{1}{D_1 ... D_n} = \int_0^1 \prod_{i=1}^n dx_i \delta(\sum_i x_i - 1) \frac{(n-1)!}{(x_1 D_1 + ... + x_n D_n)^n} \tag{235}$$

This proves eq.(233).

Working out D of that equation we find that it has a linear term in the loop momentum k. We get rid of that by the shift $k^\mu = l^\mu - (yq - zp)^\mu$ resulting in

$$\begin{aligned} D &= l^2 - \Delta + i\epsilon \\ \Delta &= -xyq^2 + (1-z)^2 m^2 \end{aligned} \tag{236}$$

We would now be able to do the loop integral

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{(l^2 - \Delta + i\epsilon)^3}$$

(after Wick rotation) using our master formula eq.(165). But unfortunately we have a complicated numerator to worry about:

$$\text{Numerator } = \Omega^\nu [(\not{k} + \not{q}) + m] \Omega^\mu [\; \not{k} + m] \Omega_\nu \tag{237}$$

The reduction of that is the most boring part of the calculation. Clearly we are going to have terms (i) quadratic in l, (ii) linear in l – those vanish by symmetry under $l^\mu \to -l^\mu$, and (iii) terms independent of l.

The *quadratic* term in l is the easiest to deal with, but in fact will not contribute to the infrared problem we have in mind. Nevertheless for completeness we also treat that. Writing

$$\Omega^\nu \; \not{l} \Omega^\mu \; \not{l} \Omega_\nu = \Omega^\nu \Omega^\alpha \Omega^\mu \Omega^\beta \Omega_\nu l_\alpha l_\beta \tag{238}$$

We see that we must understand how to do the integral

$$\int \frac{d^d l}{(2\pi)^d} \frac{l_\alpha l_\beta}{D^3}$$

This integral is a 2nd rank Lorentz tensor and in fact completely invariant under Lorentz transformations. Hence it must be proportional to $\eta_{\alpha\beta}$. Contracting with $\eta^{\alpha\beta}$ the coefficient is easily found:

$$\int \frac{d^d l}{(2\pi)^d} \frac{l_\alpha l_\beta}{D^3} = \frac{1}{d} \eta_{\alpha\beta} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{D^3} \tag{239}$$

and that integral we can do by our rules for dimensional regularization. In addition we must generalize the gamma matrix algebra, in particular we use

$$\Omega^\alpha \Omega^\mu \Omega_\alpha = (2-d) \Omega^\mu \tag{240}$$

which is easily seen to hold for d any integer (use commutation relations and $\eta^\alpha_\alpha = d$). It is a definition for d non integer. The integral is then worked out to be

$$-i \frac{(2-d)^2}{d} \Omega^\mu \Gamma\left(\frac{\epsilon}{2}\right) \Delta^{-\frac{\epsilon}{2}}$$

We expand that for small ϵ in the by now familiar way. There is a pole term in $1/\epsilon$ which we subtract by a counter term in the lagrangian, leaving a $\log \Delta(q^2, x, y, z)$, but due to the condition $F_1(0) = 1$, we further subtract a finite contribution and end up with

$$\log \frac{\Delta(q^2, x, y, z)}{\Delta(0, x, y, z)}$$

We then go to the terms *independent of* l. They are easily identified, but we want to reduce the numerator gamma algebra so as to identify contributions to F_1 and F_2 according to eq.(229). This requires a lot of boring gamma matrix algebra. One uses again and again

$$\begin{aligned} \not{p}\Omega^\mu &= -\Omega^\mu \not{p} + 2p^\mu \\ \not{p}u(p) &= mu(p) \\ x + y + z &= 1 \end{aligned} \tag{241}$$

Also one uses eq.(228) and the fact that we know that eventually (i.e. after integration over Feynman parameters) terms of the form $p'_\mu - p_\mu$ cannot contribue, so that we may replace $p'_\mu \to \frac{1}{2}(p'_\mu + p_\mu)$ etc. The result of a long calculation is

$$\begin{aligned} &\overline{u}(p')\Omega^\nu[\not{q}(1-y) + z\not{p} + m]\Omega^\mu[-y\not{q} + z\not{p} + m]\Omega_\nu u(p) \\ &= -2\overline{u}(p')[\Omega^\mu((1-x)(1-y)q^2 + (1-4z+z^2)m^2) \\ &+ \frac{i\sigma^{\mu\nu}q_\nu}{2m}2mz(1-z)]u(p) \end{aligned} \tag{242}$$

The loop integral is convergent, but it is still convenient to use the master formula eq.(165). The result of combining everything is

$$\begin{aligned} F_1(q^2) &= 1 + \frac{\alpha}{\pi}\int_0^1 dxdydz\delta(x+y+z-1) \\ &\times \left\{\log\frac{\Delta(0)}{\Delta(q^2)} + \frac{n^2(1-4z+z^2) + q^2(1-x)(1-y)}{m^2(1-z)^2 - q^2xy} - (q=0)\right\} \\ F_2(q^2) &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m^2z(1-z)}{m^2(1-z)^2 - q^2xy} \end{aligned} \tag{243}$$

Now we can see that the Feynman parameter integral for F_1 is (infra red) divergent. In fact in the corner $z \approx 1,\ x \approx 0 \approx y$ we may estimate

$$\begin{aligned} F_1(q^2)_{\text{div}} &\approx \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\left\{\frac{-2m^2+q^2}{m^2(1-z)^2} - \frac{-2m^2}{m^2(1-z)^2}\right\} \\ &= \frac{\alpha}{\pi}\int_0^1 dz\int_0^{1-z} dx\frac{q^2}{m^2(1-z)^2} = \frac{q^2}{m^2}\frac{\alpha}{\pi}\int_0^1 dz\frac{1}{1-z} \end{aligned} \tag{244}$$

which is log divergent. The problem may be regulated by a "photon mass"

$$\frac{1}{(p-k)^2} \to \frac{1}{(p-k)^2 + \mu^2}$$

(after Wick rotation) so that this regularization adds $z\mu^2$ to D. This modifies the expression for F_1 a little. The infra red singular part now becomes (same corner $z \approx 1,\ x \approx 0 \approx y$, slightly better estimate)

$$F_1^{IR}(q^2) = \frac{\alpha}{2\pi}\int_0^1 dz\int_0^{1-z} dx\left\{\frac{-2m^2+q^2}{m^2(1-z)^2 - q^2x(1-z-y) + \mu^2} - \frac{-2m^2}{m^2(1-z)^2 + \mu^2}\right\} \tag{245}$$

Evaluate by putting $x = (1-z)\xi, \;\; w = (1-z)$ and go to the w^2 variable. Then in the limit $\mu \to 0$ (so that we only do leading $\log \mu^2$)

$$
\begin{aligned}
F_1^{IR}(q^2) \;&=\; -\frac{\alpha}{2\pi} f_{IR}(q^2) \log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) \;+\; \text{NNL}(\mu \to 0) \\
f_{IR}(q^2) \;&=\; \int_0^1 d\xi \frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)} - 1 \\
&\sim \log\left(\frac{-q^2}{m^2}\right) \;\; \text{for } -q^2 \to \infty
\end{aligned}
\tag{246}
$$

It is possible to verify (ref. [5] p.206) that

$$
I(\vec{v}, \vec{v}') = 2 f_{IR}(q^2)
\tag{247}
$$

with I defined in eq.(216):

$$
\begin{aligned}
\int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{1}{(\hat{k}\cdot p)^2} \;&=\; \frac{1}{2}\int_{-1}^{1} d\cos\theta \frac{1}{[p^0 - |\vec{p}|\cos\theta]^2} = \frac{1}{2|\vec{p}|}\left(\frac{1}{p^0 - |\vec{p}|} - \frac{1}{p^0 + |\vec{p}|}\right) \\
&=\; \frac{1}{p^2} = \frac{1}{m^2}
\end{aligned}
\tag{248}
$$

For the first term in the definition of I in eq.(216) we use the Feynman parameter trick for two factors in the denominator eq.(175)

$$
\int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{2pp'}{(\hat{k}p')(\hat{k}p)} = \int_0^1 \; d\xi \int \frac{d\Omega_{\hat{k}}}{4\pi} \frac{2pp'}{[\xi \hat{k}p' + (1-\xi)\hat{k}p]^2}
\tag{249}
$$

and then do the integral over solid angle as above. Then eq.(247) is easy to verify.

If for simplicity we concentrate on the leading log limit where $-q^2 \gg m^2$, then

$$
\begin{aligned}
F_1(q^2) \;&=\; 1 - \frac{\alpha}{2\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right) \;+\; \text{NNL} \\
\frac{d\sigma}{d\Omega}(p \to p') \;&=\; \frac{d\sigma}{d\Omega}_0 \left[1 - \frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{\mu^2}\right)\right]
\end{aligned}
\tag{250}
$$

where we worked out the interference between the lowest order diagram and the vertex correction diagram to the scattering of a single electron without bremsstrahlung. The result is evidently infrared singular for $\mu \to 0$.

4.3.2 Resolution of the infrared problem

For the bremsstrahlung process we found in the same limit eq.(224)

$$
d\sigma(p \to p' + \varOmega) = d\sigma_0(p \to p') \frac{\alpha}{\pi} \log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right)
\tag{251}
$$

Of course the sum of the two is independent of μ^2. But we should not really consider the sum. Rather we should make an appropriate sum depending on the energy resolution of the photon detector used in the experiment. When measuring the cross section for scattering of the electron *without* an associated bremsstrahlung photon, in reality we only measure the inclusive cross section for the situation where possible bremsstrahlung photons have an energy lower than some E_{max}, the cut off of the detector. Hence we should work out

$$d\sigma(p \to p' + \Omega; E_{max}) = d\sigma_0(p \to p') \frac{\alpha}{2\pi} \int_{\mu}^{E_{max}} \frac{dE_\Omega}{E_\Omega} I(\vec{v}, \vec{v}') \tag{252}$$

with

$$I(\vec{v}, \vec{v}') \sim 2 \log\left(\frac{-q^2}{m^2}\right) \tag{253}$$

in the leading log approximation. Now adding the relevant contributions we obtain (leading log)

$$d\sigma(p \to p'; \text{measured}) = d\sigma_0(p \to p')[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{E_{max}^2}\right) \log\left(\frac{-q^2}{m^2}\right)] \tag{254}$$

This is acceptably independent of μ^2, it is infrared finite. That fact works also outside the leading log approximation due to the identity eq.(247).

We have solved the problem we set out to solve completely to first order in α. The solution is still not totally satisfactory from a physical point of view, perhaps. When E_{max} is very small (for a very good detector), the order α correction term becomes large, so we should go to higher orders in α. Indeed from the classical calculation we expect that there will always be infinitely many soft photons emitted, corresponding to arbitrarily high orders in α. It is of course not possible to carry out a complete treatment to any order in α, but it is possible to do so in the approximation of large logarithms

$$\log\left(\frac{-q^2}{E_{max}^2}\right), \log\left(\frac{-q^2}{m^2}\right)$$

The result is that the corrections exponentiate, resulting in the so called Sudakov form factor

$$\begin{aligned} & 1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{E_{max}^2}\right) \log\left(\frac{-q^2}{m^2}\right) + \ldots \\ \to\ & \exp\{-\frac{\alpha}{\pi} \log\left(\frac{-q^2}{E_{max}^2}\right) \log\left(\frac{-q^2}{m^2}\right)\} \end{aligned} \tag{255}$$

This vanishes in the limit $E_{max} \to 0$. That is a physically sensible result: The probability that there be absolutely no associated photon emission in a scattering process for a charged particle, is zero. The leading log calculation involves considering all possible diagrams of the form indicated in fig. 20.

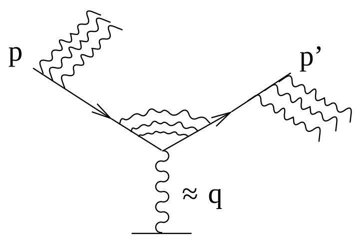


Fig. 20: Generic Feynman diagram contributing to the leading log approximation resulting among other things in the Sudakov form factor.

4.3.3 Exercise: The electron anomalous moment

To lowest order the electron has a magnetic moment

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S} \tag{256}$$

with a g factor of 2, the famous Dirac result. QED corrects that result. Huge calculations going to 3'd order in α has provided a result in spectacular agreement with the equally impressive experimental result of

$$\frac{1}{2}(g-2) = 0.001159652193(10) \tag{257}$$

From eq.(243) we may already derive the very interesting first correction to first order in α. We just need to understand why $g = 2(F_1(0) + F_2(0)$.

The matrix element

$$iT = -ie\overline{u}(p')(\Omega^\mu F_1 + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2)u(p)A_\mu(q) \tag{258}$$

we may regard as the Born approximation to the energy described by an effective hamiltonian. Using the explicit realization eq.(49) of the gamma matrices and of the spinor solutions

$$u_s(p) = \sqrt{p^0+m}\begin{bmatrix}\chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0+m}\chi_s\end{bmatrix} \quad \chi_{\pm\frac{1}{2}} = \begin{cases}\begin{pmatrix}1\\0\end{pmatrix} \\ \\ \begin{pmatrix}0\\1\end{pmatrix}\end{cases} \tag{259}$$

we find for small q

$$iT_{fi} = -i(2m)e\chi^\dagger\left[-\frac{1}{2m}\sigma^k[F_1(0)+F_2(0)]\right]\chi B^k(q) \tag{260}$$

with magnetic field

$$B^k(q) = -i\epsilon^{ijk}q^iA^j(q) \tag{261}$$

We compare that with the magnetic energy

$$-\vec{\mu}\cdot\vec{B}$$

using $\vec{S}=\frac{1}{2}\vec{\sigma}$, and read off

$$g = 2(F_1(0)+F_2(0)) \tag{262}$$

It is now quite trivial to do the integral over Feynman parameters in eq.(243) with the result

$$F_2(0) = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2) = 0.0011614 + \mathcal{O}(\alpha) \tag{263}$$

which is already in impressive agreement wit experiment (it was first derived by Schwinger and by Feynman).

5 LECTURE 5. HOW COUPLING CONSTANTS RUN

In lecture 3 we saw how it came about that ϕ^4 theory developed a running coupling constant (1 loop)

$$\lambda_{\phi^4}(q^2) = -\frac{32\pi}{3\log(q^2/\Lambda^2_{\phi^4})} \tag{264}$$

In this lecture we would like to understand in some detail why the corresponding expression in QCD (generalized to N_C number of colours) has the form (1 loop)

$$\frac{g^2(q^2)}{4\pi} = \alpha_{QCD}(q^2) = +\frac{12\pi}{(11N_C - 2N_f)\log(q^2/\Lambda^2_{QCD})} \tag{265}$$

Here N_f is the number of "open" quark flavours, meaning the number for which the quark masses are much less than $\sqrt{q^2}$. We wish to understand the all important change of sign between ϕ^4 and QCD, the latter being asymptotically free: when $q^2 \rightarrow +\infty$ the log in the denominator becomes positive and very large and the (positive) strong fine structure constant goes to zero. Further, what is the origin of the numbers $+11$ and -2 in front of N_C and N_f? Also, if we generalize QCD to SUSY QCD, then

$$\begin{aligned} 11N_C &\rightarrow 9N_C \\ -2N_f &\rightarrow -3N_f \end{aligned} \tag{266}$$

How does that come about? And indeed, will it be possible to develop some sort of physical picture for all that?

In lecture 3 we obtained the result by studying Feynman 1-loop diagrams. The same procedure will be possible here and probably represents the technically safest way of doing the calculation. However, that way leads to a very large amount of rather boring calculations, and also one does not gain much physical intuition from it. Instead we shall use a different method. We shall calculate what amounts to the so-called effective potential in a certain classical background of constant colour magnetic field. Such calculations also provide the result with much less effort, and they give a certain amount of idea as to the physical mechanism. In this lecture we shall follow the treatment of N.K. Nielsen [9]. It is precisely designed to provide a pedagogical presentation while sacrificing only a little in technical precision. In addition it provides an interesting unifying formula for various contributions to the way the coupling constant runs.

5.1 The idea and the result

The running of the coupling constant is associated with the occurence in a quantum field theory of a scale dependent polarizability of the vacuum. Vacuum fluctuations associated with the zero point motion of the infinitely many harmonic oscillators, will give rise to pair creation of quanta that will tend to screen or antiscreen the measured colour charge of a test quark. Thus, for close separation between two test quarks of colour charges q_1, q_2 where forces are approximately coulombian, the potential between them has the form

$$V(r) = \frac{q_1 q_2}{4\pi\epsilon(r) r} \tag{267}$$

where $\epsilon(r)$ is a dielectric "constant" that will turn out to be dependent on distance, or in Fourier space, on momentum (squared). It is the "running" of ϵ we would want to understand. It provides for a "running charge" of $q_1^2(r) \sim q_1^2/\epsilon(r)$.

However, rather than subjecting the vacuum to an electric field it will turn out to be more convenient to test the vacuum with a constant colour magnetic field. That way we may learn about the magnetic permeability of the vacuum, μ, and since the vacuum is Lorentz invariant, we know that (Maxwell taught us that)

$$\mu\epsilon = c^2 = 1 \tag{268}$$

Thus we have

$$\begin{array}{ll} \text{screening :} & \epsilon > 1,\ \mu < 1 \ \text{QED, } \phi^4 \\ \text{antiscreening:} & \epsilon < 1,\ \mu > 1 \ \text{QCD, asymptotic freedom} \end{array} \tag{269}$$

We may learn about the permeability by studying the shift in energy due to the presence of the magnetic field. In fact the energy density is

$$\begin{aligned}\text{Energy density} &= -\frac{1}{2}4\pi\chi(H)H^2 \\ \mu &\equiv 1+4\pi\chi \end{aligned} \tag{270}$$

Here χ is the magnetic susceptibility which we anticipate will be scale dependent like everything else, and the only scale in the problem here will be the magnetic field itself, H.

In fact let us make a short cut in the discussion. The polarizability implies that in fact the effective measured field also "runs". However, the combination

$$gH$$

does not run! It is a renormalization group invariant. We shall not prove this fact, but it is very plausible that this should be so. Remember that we introduced the gauge covariant derivative $D_\mu = \partial_\mu + \mathcal{A}_\mu$ in lecture 2, and that

$$\mathcal{A}_\mu = -igA^a_\mu \mathcal{T}^a$$

The statement is that $\mathcal{A}_\mu$ does not run. Only the individual factors g and A^a_μ do. Of course ∂_μ cannot run, so the gauge invariance would become a mess if $\mathcal{A}_\mu$ started to run.

Accepting that gH is a non-running renormalization group invariant quantity, we may write for the well known energy

$$\text{Energy} = \frac{1}{2}VH^2 \to \frac{1}{2}V\frac{(gH)^2}{g^2(gH)} \tag{271}$$

Here V is the volume of the system. We see that if we can evaluate the energy in the presence of a colour magnetic field, including the effect of quantum fluctuations (to some approximation), then we can read off the running coupling constant from the result at the scale which is gH. Thus for QCD we shall find

$$\begin{aligned}E^{QCD}_{vac} &= \frac{1}{2}V\frac{(11N_C-2N_f)}{48\pi^2}\log\frac{|gH|}{\Lambda^2} \Rightarrow \\ \frac{g^2(gH)}{4\pi} &= \frac{12\pi}{(11N_C-N_f)\log\frac{|gH|}{\Lambda^2}}\end{aligned} \tag{272}$$

which of course is just the result we would like to understand. Notice that gH has the same dimension as a q^2.

In actual fact we shall be able to do somewhat better than that. We shall establish a master formula, which will describe the contribution of various kinds of quanta to the effect of the quantum fluctuations. Namely we shall consider 3 kinds of massless quanta of spin s and with helicity $s_3 = \pm s$ for $s \neq 0$. We shall consider scalars of spin 0, fermions of spin $\frac{1}{2}$ and gauge bosons of spin 1. The contribution of *a pair of particle-antiparticle-quanta* will contribute the following

$$\Delta E_{vac} = -(-)^{2s}\sum_{s_3}\left(\frac{s_3^2}{2}-\frac{1}{24}\right)V\frac{(gH)^2}{4\pi^2}q^2\log\frac{\Lambda^2}{gH} \tag{273}$$

Here $\pm q$ denotes the strength by which the quanta in question couple to the background field H in units of g. We notice the following:

- When spin is presents it plays a dominant role – via a magnetic moment coupling. Paramagnetism is dominating over diamagnetism.
- There is a crucial difference in sign for bosons and fermions $(-)^{2s}$.

- There is fixed diamagnetic contribution independent of spin. It has the opposite sign to the paramagnetic contribution.

It is instructive to work out the numerical factors for the 3 kinds of quanta:

$$(-)^{2s}\sum_{s_3}\left(\frac{s_3^2}{2}-\frac{1}{24}\right)=\begin{cases}-\frac{1}{24} & \text{scalars}\\ -\frac{1}{6} & \text{fermions}\\ +\frac{11}{12} & \text{gauge bosons}\end{cases} \tag{274}$$

We shall also need the sum over a multiplet (R) of $SU(N_C)$ of q^2, the various strengths by which quanta-antiquanta pairs couple to the background. It will turn out to be the case that

$$\sum_{\text{multiplet}(R)} q^2 = r(R)\cdot C_2(R) \tag{275}$$

where $C_2(R)$ is the quadratic Casimir of the representation R (in a suitable normalization) and $r(R)$ is $\frac{1}{2}$ for real representations and 1 for complex representations. We shall just show for $SU(N_C)$ that

$$\sum_{\text{multiplet}} q^2 = \begin{cases}\frac{1}{2} & \text{fundamental rep.}\\ \frac{N_C}{2} & \text{adjoint rep.}\end{cases} \tag{276}$$

It is now immediate to see that all that reproduces the QCD result eq.(265). Also it is interesting to see how things are modified in the case of supersymmetry. Thus the effect of adding gluinos (spin $\frac{1}{2}$ fermions) to gluons is to carry out the replacement

$$\frac{11}{12}\frac{N_C}{2}\to\left(\frac{11}{12}-\frac{1}{6}\right)\frac{N_C}{2}=\frac{9}{2}\frac{N_C}{2} \tag{277}$$

Similarly, adding squarks has the effect

$$\left(-\frac{1}{6}\right)\frac{N_f}{2}\to\left(-\frac{1}{6}-\frac{2}{24}\right)\frac{N_f}{2}=-\frac{N_f}{8} \tag{278}$$

Both of these fit the claim in eq.(266). Notice that according to supersymmetry, the SUSY partners have the same number of degrees of freedom as the original ones. Thus 2 gluon-helicities correspond to 2 gluino helicities, characteristic of so-called Majorana fermions. However, as emphasized we shall use a counting with quanta and anti-quanta taken together. Thus we shall count gluons two at a time, rather like the W^+ and W^- particle-antiparticle gauge bosons. Therefore it is correct, as done here, to use the contribution strictly worked out for Dirac fermions. Similarly since quarks have 2 spin states, SUSY needs 2 squarks (in addition to the anti-squarks), thus the factor of 2 on scalars above. Notice also that the master formula, powerful as it is, is not immediately applicable to the electroweak case, since there one uses chiral fermions, whereas the derivation we shall give below is based on Dirac-fermions.

Thus the master formula is capable of giving us a physical picture of how the various quanta contribute to the polarizability of the vacuum-medium. The rest of the lecture is devoted to deriving the master formula, eq.(273) together with eq.(276).

5.2 The contribution of massless scalars

Scalars will provide a contribution independent of spin, which we shall find again in the following two subsections about fermions and gauge bosons.

We consider a particle-antiparticle pair of scalars. Before an external magnetic field is imposed, they are described by "harmonic oscillators" of creation and annihilation operators as explained in lecture 2:

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'} = [b_{\vec{k}}, b^\dagger_{\vec{k}'}] \tag{279}$$

The energy spectrum of these 2 harmonic oscillators is very well known

$$E^+_{n,\vec{k}} = \omega_{\vec{k}}(n + \frac{1}{2}) = E^-_{n,\vec{k}} \tag{280}$$

where $E^\pm$ refer to the particle and the anti particle. n is the occupation number, and for massless fields (which are the only ones we shall consider)

$$\omega_{\vec{k}} = |\vec{k}| \tag{281}$$

obtained as the positive energy solution of the massless Klein-Gordon equation

$$\omega^2_{\vec{k}} - \vec{k}^2 = 0$$

Thus we see that the theory provides for the hugely divergent vacuum energy of

$$\text{vacuum energy} = \sum_{\vec{k}} (\frac{1}{2}\omega_{\vec{k}} + \frac{1}{2}\omega_{\vec{k}}) = \sum_{\vec{k}} \omega_{\vec{k}} \tag{282}$$

The Klein-Gordon equation

$$\begin{aligned} \partial_\mu \partial^\mu \phi &= 0 \\ -\Delta\phi = -\partial_t^2 \phi &= E^2 \phi \end{aligned} \tag{283}$$

is modified in the presence of a magnetic field by

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu - igqA_\mu \tag{284}$$

where g is the gauge coupling constant, and $\pm q$ is the strength in units of g by which the two quanta couple to the background A_μ. We introduce the notation

$$gq \equiv e \tag{285}$$

We may take

$$A_\mu = (0, 0, x_1 H, 0) \Rightarrow \partial_1 A_2 = F_{12} = H \tag{286}$$

describing a magnetic field along the z axis. Also in this gauge $\partial_\mu A^\mu = 0$. So now we merely have to extract ∂_t^2 from the equation of motion

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi = 0$$

The result is that the new energy is determined by the eigenvalue equation

$$(\Delta - e^2 H^2 x_1^2 - 2iex_1\partial_2)\phi = -E^2\phi \tag{287}$$

Physically this equation describes levels analogous to the Landau levels in a non relativistic situation. We find the spectrum by introducing the following ansatz for the eigenfunctions

$$\phi(x) = e^{i(k_2 x_2 + k_3 x_3)} \phi_n(x_1 - \frac{k_2}{eH}) \tag{288}$$

This brings the eigenvalue equation to the form

$$(\partial_x^2 - e^2H^2x^2)\phi_{n,k_3}(x) = -(E_{n,k_3}^2 - k_3^2)\phi_{n,k_3}(x)$$
$$x \equiv x_1 - \frac{k_2}{eH} \tag{289}$$

This equation is nothing else that the non relativistic Schrödinger equation for a harmonic oscillator. Thus we find the spectrum immediately

$$E_{n,k_3}^2 = k_3^2 + 2eH(n + \frac{1}{2}) \tag{290}$$

We see the translational motion along the 3 axis, and the spiralling motion of the quanta manifested in the harmonic oscillator levels. We have to remember that we have been dealing with a shifted harmonic oscillator, the shift being given by

$$(x_1)_{\text{shift}} = -k_2/eH \tag{291}$$

Thus, for each n, k_2, k_3 there is a zero mode we have to include in the total vacuum energy

$$E_{vac}(\text{scalars}) = \sum_{n,k_2,k_3} E_{n,k_3} \tag{292}$$

We shall have to come back to understanding how to make sense out of that sum. Notice in particular, that the spectrum does not depend explicitly on k_2 after we have performed the shift eq.(291). The details here depend on the gauge choice we have made.

5.3 Massless Dirac field

Consider "quarks" in the fundamental representation of $SU(N_C)$. The set of generators are represented by $N_C \times N_C$ matrices $\{\mathcal{T}^a\}$, $a = 1, 2, ..., N_C^2 - 1$ and normalized as

$$Tr(\mathcal{T}^a\mathcal{T}^b) = \frac{1}{2}\delta^{ab} \tag{293}$$

Let us fist concentrate on the $N_C - 1$ commuting diagonal matrices. We can satisfy the normalization eq.(293) when we use the set

$$\frac{1}{2}\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix};\ \ \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix};\ \cdots$$

$$\mathcal{T} = \frac{1}{\sqrt{2N_C(N_C-1)}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & -(N_C-1) \end{pmatrix}$$

We shall not need the non-diagonals for now. We decide to take the gauge field along the direction of $\mathcal{T}$, our short hand name for the last diagonal matrix, in algebra space. Thus we take the background gauge field to be

$$\mathcal{A}_\mu = -ig\mathcal{T}A_\mu \tag{294}$$

with A_μ just as before for scalars eq.(286). Thus we see that the N_C different quarks couple with strengths or colour charges

$$qg = \frac{g}{\sqrt{2N_C(N_C-1)}}\{1,1,...,1,-(N_C-1)\} \tag{295}$$

so that obviously

$$\sum_{\text{fundamental multiplet}} q^2 = \mathrm{tr}(\mathcal{T}^2) = \frac{1}{2} \tag{296}$$

as promised in eq.(276).

In the present fermionic case we have two sets of fermionic creation and annihilation operators for the quark and the antiquark. The crucial difference this time is that the vacuum zero point energy is *negative* for fermions. The easiest way to understand this is to think in terms of Dirac's picture of what a vacuum is: for a pair of fermionic oscillators it is the state where the negative energy solution is occupied and the positive energy solution is not. The result is that when we sum over the energy spectrum to obtain the contribution to the vacuum energy, we must have an extra minus sign in front.

We now work out the spectrum taking into account the background gauge field. The Dirac equation in the presence of the background is $\not{\mathcal{D}}\psi = 0$. From that we derive the equations

$$\begin{aligned}
\not{\mathcal{D}}\not{\mathcal{D}}\psi &= 0 \\
\mathcal{D}_\mu\mathcal{D}_\nu \varOmega^\mu\varOmega^\nu\psi &= 0 \text{ or} \\
(\frac{1}{2}[\mathcal{D}_\mu,\mathcal{D}_\nu] + \frac{1}{2}\{\mathcal{D}_\mu,\mathcal{D}_\nu\})(\frac{1}{2}[\varOmega^\mu,\varOmega^\nu] + \frac{1}{2}\{\varOmega^\mu,\varOmega^\nu\})\psi & \\
(\frac{1}{4}[\mathcal{D}_\mu,\mathcal{D}_\nu][\varOmega^\mu,\varOmega^\nu] + \frac{1}{4}\{\mathcal{D}_\mu,\mathcal{D}_\nu\}\{\varOmega^\mu,\varOmega^\nu\})\psi &= (-\frac{1}{4}igqF_{\mu\nu}[\varOmega^\mu,\varOmega^\nu] + \mathcal{D}^2)\psi \\
&= 0
\end{aligned} \tag{297}$$

The last term is exactly what we met in the scalar case and we know the spectrum for that. The first term is the magnetic moment coupling. For the background chosen we only get terms for $(\mu,\nu) = (1,2)$ or $(2,1)$, and the gamma matrix commutator then involves $2\sigma_3$, so that altogether we obtain from that term ($s_3 = \frac{1}{2}\sigma_3$)

$$-2eHs_3$$

and the eigenvalue equation

$$(\Delta - e^2H^2x_1^2 - 2ieHx_1\partial_2 + 2eHs_3)\phi = 0 \tag{298}$$

Based on the scalar result we may now immediately write down the spectrum

$$E^2_{n,k_2,k_3,s_3} = k_3^2 + 2eH(n+\frac{1}{2}) + 2eHs_3 \tag{299}$$

Here $s_3 = \pm\frac{1}{2}$. As we shall see in the next subsection, we obtain exactly the same formula for gauge bosons, only there $s_3 = \pm 1$.

5.4 The contribution from gauge bosons

When we consider the effect of quantum fluctuation of the gauge field itself, we must distinguish between the background part $\mathcal{A}_\mu^{(b)}$ and the fluctuating quantum part $\mathcal{A}_\mu^{(q)}$ of the gauge field. Correspondingly we write

$$\mathcal{A}_\mu = \mathcal{A}_\mu^{(b)} + \mathcal{A}_\mu^{(q)} \tag{300}$$

The equations of motion for a non-Abelian gauge field may be derived from lecture 2 and are (exercise)

$$\begin{aligned} \mathcal{D}_\mu \mathcal{F}^{\mu\nu} &= 0 \quad \text{meaning} \\ \partial_\mu \mathcal{F}^{\mu\nu} + [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] &= 0 \end{aligned} \tag{301}$$

where as before

$$\mathcal{A}_\mu^{(b)} = -ig\mathcal{T} A_\mu$$

whereas

$$\mathcal{A}_\mu^{(q)} = -ig\mathcal{T}^a A_\mu^a \tag{302}$$

sum over a implied. We see from the equation of motion that if $[\mathcal{T}, \mathcal{T}^a] = 0$ then the quantum fluctuation labelled a will be neutral with respect to the background, and will not contribute to the *shift* in vacuum energy due to the background. This in particular will be true for all those $\mathcal{T}^a$'s that are diagonal. For the non diagonal ones we may choose matrices that generalize the two non-diagonal Pauli matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{303}$$

except we put the 1's and $\pm i$'s at arbitrary positions (i, j) and (j, i). However, just as for $SU(2)$ it is often more convenient to work with a different (complexified) basis in which we take linear combinations of those two, so that our general basis of non-diagonals consists simply of matrices with 0's everywhere except for a 1 at (row,column) $= (i, j)$ $(i \neq j)$. In general these matrices too will commute with the matrix $\mathcal{T}$ eq.(294). The only exceptions are $i = N_C$ or $j = N_C$, for which we have $N_C - 1$ of each. We denote those

$$\mathcal{T}^\alpha$$

with $\alpha = 1, 2, ..., N_C$ corresponding to

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{304}$$

and with $\mathcal{T}^{-\alpha} \equiv (\mathcal{T}^\alpha)^\dagger$. It is easy to work out that

$$[\mathcal{T}, \mathcal{T}^\alpha] = \sqrt{\frac{N_C}{2(N_C - 1)}}\, \mathcal{T}^\alpha \tag{305}$$

showing that these $N_C - 1$ gauge bosons have charge $q = \sqrt{\frac{N_C}{2(N_C-1)}}$ with respect to the background. Similarly the anti gauge bosons corresponding to the $\mathcal{T}^{-\alpha}$'s have the opposite charge. Hence we find

$$\sum_{\text{adjoint}} q^2 = 2(N_C - 1)\frac{N_C}{2(N_C - 1)} = N_C \tag{306}$$

Here the factor 2 comes because the sum over the full multiplet involves both the $\mathcal{T}^\alpha$'s and the $\mathcal{T}^{-\alpha}$'s. However, these represent gluons and antigluons, and in our notation we should only count the pair as one. Hence we obtain the result

$$\frac{1}{2}N_C$$

as promised in eq.(276).

We may now write for small quantum fluctuations

$$\mathcal{F}^{\mu\nu} = -ig\mathcal{T}F^{\mu\nu} + \sum_{\alpha}(F_{\alpha}^{\mu\nu}\mathcal{T}^{\alpha} + F_{-\alpha}^{\mu\nu}\mathcal{T}^{-\alpha}) + ... \tag{307}$$

where the dots represent terms commuting with the background, and where

$$\begin{aligned} F^{\mu\nu} &= \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} + \mathcal{O}((A_{\alpha})^2) \\ F_{\alpha}^{\mu\nu} &= \partial^{\mu}A_{\alpha}^{\nu} - \partial^{\nu}A_{\alpha}^{\mu} - ie(A^{\mu}A_{\alpha}^{\nu} - A^{\nu}A_{\alpha}^{\mu}) \\ e &\equiv gq = g\sqrt{\frac{N_C}{2(N_C-1)}} \end{aligned} \tag{308}$$

We now impose the *background gauge condition*

$$\partial^{\mu}\mathcal{A}_{\mu}^{(q)} + [\mathcal{A}_{\mu}^{(b)}, \mathcal{A}^{(q)\mu}] = 0 \tag{309}$$

One then finds after a few lines of algebra that the equations of motion become

$$\{(\partial_t^2 - \Delta) + 2iex_1H\partial_2 + e^2H^2x_1^2\}A_{\nu}^{\alpha} - 2ieF_{\nu}{}^{\mu}A_{\mu}^{\alpha} = 0 \tag{310}$$

The first piece we recognize from the scalar problem. The last piece is the (colour) magnetic moment coupling. It is not quite an eigenvalue equation, but we make it into one by considering the combinations

$$A_1^{\alpha} \pm iA_2^{\alpha}$$

Then the spin part gets exactly the same form as in the Dirac case with $s_3 = \pm 1$ as promised.

There is a technical point to worry about. How about the 2 degrees of freedom without any magnetic moment coupling? The correct treatment of these would be in terms of the Fadeev-Popov ghost contribution and by showing that in the present case, with the precise background gauge condition we have imposed, these ghost contributions cancel the extra gauge boson contributions. We shall not do that here, however, but merely go on with the contribution from the two "physical" gauge bosons with the two physical polarization states.

5.5 The vacuum energy

We have seen that a formal expression for the vacuum energy in all cases may be written

$$E_{vac} = (-)^{2s}\sum_{k_2,k_3,n,s_3}\{k_3^2 + 2eH(n+\frac{1}{2}+s_3)\}^{\frac{1}{2}} \tag{311}$$

We shall now try to make sense of this rather badly diverging expression. In particular we shall want to understand the *H-dependence*, not worrying too much about the infinite vacuum energy in the absence of a background. We shall be looking for a behaviour of the form

$$(gH)^2\log(gH/\Lambda^2)$$

with Λ a cut-off. That cut-off will be converted to the finite QCD scale Λ_{QCD} as a result of renormalization, more or less along the lines of lecture 3. But that part we shall not pursue. We shall be satisfied with finding the gH dependence and with working out in particular the numerical coefficient corresponding to eq.(273).

Let us first consider the sum over k_2, k_3. There is a trivial divergence coming from the infinite size of the world. We regulate the problem by introducing a quantization volume $V = L^3$. The number of quantum states with momentum in the interval Δk is then $\frac{L}{2\pi}\Delta k$ in each dimension. There is a seeming mystery since we have no k_2 dependence. However, remembering that we were obliged to shift the position of the harmonic oscillator by the amount k_2/eH eq.(291), we see that we should demand

$$0 \leq \frac{k_2}{eH} \leq L \Rightarrow \sum_{k_2} \rightarrow \frac{L}{2\pi}(eHL) \tag{312}$$

Thus

$$\sum_{k_2,k_3} \rightarrow \frac{V}{(2\pi)^2} eH \int dk_3 \tag{313}$$

and

$$E_{vac} = (-)^{2s} \frac{V}{(2\pi)^2} eH \sum_{n,s_3} \int dk_3 \{k_3^2 + 2eH(n + \frac{1}{2} + s_3)\}^{\frac{1}{2}} \tag{314}$$

The sum over n and the integral over k_3 we now regulate by an energy cut-off

$$E^2 < \Lambda^2$$

Then

$$0 \leq n < \frac{\Lambda^2}{2eH} \equiv n_\Lambda;\ k_3^2 < \Lambda^2 - 2eH(n + \frac{1}{2} + s_3) \equiv k_\Lambda \sim \Lambda^2$$

Consider doing the sum over n. The first idea would consist in replacing the sum by an integral

$$\sum_n \rightarrow \int dn$$

But then a shift of variable $n \rightarrow eHn$ shows that the result of this approximation would be completely independent of H! In other words, that approximation will precisely give us the vacuum energy in the absence of a background field. So, we are exactly interested in the *corrections* to replacing the sum by an integral. This correction is provided by a simple rule due to Euler:

$$\sum_{n=n_1}^{n_2-1} f(n + \frac{1}{2}) = \int_{n_1}^{n_2} f(x)dx - \frac{1}{24} f'(x)|_{n_1}^{n_2} + ... \tag{315}$$

Proof:

$$\begin{aligned}
\int_n^{n+1} dx f(x) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \{f(n + \frac{1}{2}) + xf'(n + \frac{1}{2}) + \frac{x^2}{2} f''(n + \frac{1}{2}) + ...\}dx \\
&= f(n + \frac{1}{2}) + 0 + \frac{1}{2} \cdot \frac{1}{3} \cdot 2 \cdot \frac{1}{8} f''(n + \frac{1}{2}) \\
&\Rightarrow f(n + \frac{1}{2}) = \int_n^{n+1} f(x)dx - \frac{1}{24} f''(n + \frac{1}{2}) \Rightarrow \\
\sum_{n_1}^{n_2-1} f(n + \frac{1}{2}) &= \int_{n_1}^{n_2} f(x)dx - \frac{1}{24} \sum_{n_1}^{n_2-1} f''(n + \frac{1}{2})
\end{aligned} \tag{316}$$

Here the last sum may be replaced by the integral since it is already a correction. This proves eq.(315). The usefulness of Euler's rule depends on $f(n)$ being sufficiently smooth that higher order terms may be neglected.

We may now write for the interesting (H-dependent) part of the vacuum energy

$$\begin{aligned}
E_{vac} &= (-)^{2s} \sum_{s_3} \left[\sum_{n=0}^{n_\Lambda} f(n + \frac{1}{2} - s_3) - \int_0^{n_\Lambda} dn f(n + \frac{1}{2} - s_3) \right] \\
f(x) &\equiv \frac{V}{4\pi^2}(eH)2 \cdot \int_0^{k_\Lambda} dk_3 [k_3^2 + 2eHx]^{\frac{1}{2}}
\end{aligned} \tag{317}$$

For technical reasons we want to split the sum

$$\sum_0^{n_\Lambda} = \sum_0^N + \sum_N^{n_\Lambda}$$

with an N large enough that we may use Euler's summation formula on the last part. The final result of course must be independent of N. Thus

$$E_{vac} = (-)^{2s} \sum_{s_3} \sum_{n=N}^{n_\Lambda} \left[f(n+\frac{1}{2}) - s_3 f'(n+\frac{1}{2}) + \frac{s_3^2}{2} f''(n+\frac{1}{2}) + ... \right] + \Phi(eH, N) \tag{318}$$

Here $\Phi(eH, N)$ will be independent of Λ and therefore (for dimensional reasons) proportional to $(eH)^2$ without any $\log$ contribution. In other words Φ will be a non-leading-log contribution which we may ignore. The term linear in s_3 will vanish upon sum over s_3. The sum over f'' terms may be replaced by an integral and converts to an f' term at the limits, just like the correction term in the Euler formula.

To work out f' we use the definition eq.(317) and find

$$\begin{aligned} f'(x) &= \frac{V}{4\pi^2} 2(eH)^2 \int_0^{k_\Lambda} dk_3 \frac{1}{\sqrt{k_3^2 + 2eHx}} \\ &\sim \frac{V}{4\pi^2} 2(eH)^2 \log \frac{\Lambda}{\sqrt{eHx}} \\ f'(x)|_N^{n_\Lambda} &= \frac{(eH)^2 V}{4\pi^2} \left[\log \frac{\Lambda^2}{2eHx} \right]_N^{n_\Lambda} \\ &= -\frac{(eH)^2 V}{4\pi^2} \log \frac{\Lambda^2}{2eHN} \end{aligned} \tag{319}$$

Here we may ignore the N in the argument of the logarithm to leading log order. Putting the pieces together, we exactly reproduce the master formula eq.(273).

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